MAKING SENSE

Teaching and Learning Mathematics with Understanding

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Foreword

Almost all, who have ever fully understood arithmetic, have been obliged to learn it over again in their own way.

Warren Colburn

When I read old arithmetic textbooks such as Colburn's, I wonder what would have happened if we had listened to the advice of those who, through the ages, have recommended that mathematics should make sense to students. The tiny book of Colburn was hailed as the most valuable school book in the country. One user testified, "I find that those children introduced to arithmetic by it, have a clearer *understanding* of the operations than those who use any other introduction whatever. I believe that the universal adoption of it as elementary work would increase the intelligence of all the children in the country" (Colburn 1849, back cover, emphasis added).

Although it was a popular text, I doubt if the author's claims, made in the key to a later edition of the text, were realized:

> Instructers [*sic*] who may never have attended to fractions need not be afraid to undertake to teach this book. The author flatters himself that the principles are so illustrated and the processes are made so simple, that any one, who shall undertake to teach it, will find himself familiar with fractions before he is aware of it, although he knewth nothing of them before; and that every one will acquire a facility in solving questions which he never before possessed. (Colburn 1849, 141)

We still have students and teachers who have not developed those understandings and, furthermore, some who do not believe that they can or need to develop understanding. We all have listened to students who have come to believe that mathematics does not need to make sense. Recently, a teacher in one of my classes interviewed her students about proportional statements taken from Lamon (1995). One sixth-grade student was asked whether the statement "if one girl has three brothers, then two girls have six brothers" made sense. The student responded, "It really does not make much sense. But, we are in math class, so I guess it does in here." During one of my first years of teaching, one of my most capable students said, "In math, I do things just the opposite way from what I think they should be, and it almost always works." She was referring to the rule about a positive second derivative indicating a possible minimum, but she cited many more of her interpretations of these "nonsense" rules.

Why do we worry about making sense or understanding what we are doing? The sixth-grade student was receiving good grades, as was my college student. If we turn to the wisdom of the past, we gain much insight about understanding. The operational definition of understanding may have changed, but often the reasons given for helping students develop understanding are just as applicable today as in the past. After briefly touching on perspectives from history, we will return to the question of why we have not made more progress, and to thoughts about what this book may contribute to the pursuit of learning with understanding.

What Can We Learn from the Past About Understanding?

Throughout this century psychologists, educators, and others have been concerned with understanding. I have selected only a few of those who focused on learning relevant to mathematics to illustrate the changing views of and emphases on understanding. In their works, often interspersed in the ebb and flow of attention to understanding, we see different views of understanding and recommendations for learning with understanding.

Early in the Century

Thorndike, in psychology, and Dewey, in philosophy, both influenced education in the beginning third of this century. Dewey cautioned that the practice of teaching without understanding damaged students' ability to reflect and to make sense of what they were doing:

> Sheer imitation, dictation of steps to be taken, mechanical drills may give results most quickly and yet strengthen traits likely to be fatal to reflective power. The pupil is enjoined to do this and that specific thing, with no knowledge of any reason except that by so doing he gets his result most speedily; his mistakes are pointed out and corrected for him, he is kept at pure repetition of certain acts till they become automatic. Later, teachers wonder why the pupil reads with so little expression, and figures with so little intelligent consideration of the terms of his problem. (Dewey 1910, 51–52)

For Dewey, learning was problem solving that depended upon an individual's aims and interests. He saw the learning as emerging from experiences or problem-solving activities. The role of the teacher was to provide these experiences for the individual.

This view of learning was often contrasted and often credited with the downfall of Thorndike's connectionism. However, reaction to the progressive movement, especially the misinterpretations of incidental learning, prodded the education community to return to what more closely resembled Thorndike's principles of learning. We entered a period in which there was more emphasis on drill of small, isolated facts—exactly what Dewey was cautioning against.

The Thirties, Forties, and Early Fifties

As the pendulum continued to swing toward a more mechanistic learning, Brownell provided an alternate force as he wrote about meaningful learning:

> According to the *meaning* theory, the ultimate purpose of arithmetic instruction is the development of the ability to *think* in quantitative situations. The word "think" is used advisedly: the ability merely to perform certain operations mechanically and automatically is not enough. Children must be able to analyze real or described quantitative situations. (Brownell 1935, 28)

During the next twenty years, much of Brownell's research centered on learning with meaning and he proposed the following ten reasons to develop meaning as a student learned:

1. gives assurance of retention
2. equips him with the means to rehabilitate quickly skills that are temporarily weak
3. increases the likelihood that arithmetical ideas and skills will be used
4. contributes to ease of learning by providing a sound foundation and transferable understandings
5. reduces the amount of repetitive practice necessary to complete learning

6. safeguards him from answers that are mathematically absurd
7. encourages learning by problem solving in place of unintelligent memorization and practice
8. provides him with a versatility of attack which enables him to substitute equally effective procedures for procedures normally used but not available at the time
9. makes him relatively independent so that he faces new quantitative situations with confidence
10. presents the subject in a way which makes it worthy of respect (Brownell 1947, 263–64)

One of the strongest but least known advocates for meaning theory during this time was Wheat. In his book, *Psychology and Teaching of Arithmetic* (1937, iii), he disputed a prevalent interpretation of incidentalist beliefs that "arithmetic was part of the material world in which children live, and that it may be extracted by them in proportion to their contacts with the material world." His book also challenged the learning of separate number facts and called for learning about number ideas and relationships (185).

His comparison of learning arithmetic to taking a trip addresses both the role of the student and the role of the teacher:

> The pupil learns the way to think about the numbers of things that we call arithmetic as he explores the way. He learns the road of thinking and how to move along it as he travels the road. The speed of his movement is of minor importance as compared with the fact that, to progress, the pupil travels under his own power.
>
> In the case of the tourist, we map the route he should take and we point it out to him. In the case of the pupil, his teacher must do the same thing. While he is yet unable to recognize number questions, his teacher must point them out and make them clear. While he is yet untrained in the art of finding answers, his teacher must make clear what the finding of answers requires. His teacher must make clear what the pupil must look for and observe as he starts the unexplored part of his intellectual journey, at one place, then another along the road of number-thinking. Since his journey is intellectual, its nature and extent are determined by the objects of his attention and the ways he attends to them. (Wheat 1951, 25, 27)

Near the end of this time there was much discussion of what was meant by meaning. Van Engen (1947), in describing the development of meaning from the 1920s to the early 1940s, remarked that even Thorndike's

writings indicated support of meaning but they never clearly addressed what meaning meant. In response to the lack of definition of meaning, many of Van Engen's writings addressed this need.

Van Engen urged that "any definition of meaning should enable the teacher to abstract from each experience those specific elements which develop mathematical meanings" (64). Van Engen's quote of Einstein, "If you want to know what I mean, don't listen to what I say, but watch what I do" exemplified his own message to teachers about how to develop meaning. In summary, he recommended the following: Show the students (or let them perform) the actions implied by operations on objects. As you talk, do not expect the students to learn without observing the actions on the objects. Then help students symbolize the actions, and later generalize to larger numbers for which the actions on objects become awkward. At this stage the structure of mathematics, or the generalization of the operation, should allow the student to work symbolically.

New Math and Back to Basics

Even in Van Engen's early work we begin to see influences that would be present in the "new math" movement. Meaning of school mathematics in the modern mathematics era was derived more from the structure of mathematics than it had been previously. We can also see residues of Brownell's reasons for developing meaning in the principles put forth by Bruner (1960). He urged that the learning of mathematics should be based on the understanding of fundamentals in a structured pattern. This would ensure that details be less rapidly forgotten and when forgotten they could be more easily reconstructed when needed. He also built the case that such learning would facilitate transfer.

The behaviorist approach that was present during the new mathematics time was embraced by those who returned to the so-called basics. In the era of behavioral objectives, we heard less about meaning, thinking, and understanding partly because they were difficult to measure. If we wrote an objective stating that students should understand, we were in deep trouble. Yet even in this time, many people were still studying and struggling with the ideas of understanding. Skemp helped practitioners to consider understanding from two perspectives. In a classic article in 1976 (reprinted in Skemp 1987), he defined *relational understanding* as knowing what to do and why and *instrumental understanding* as knowing what to do or the possession of a rule and ability to use it. For example, students say, "Oh, I understand" when they remember the next step in an algorithm (instrumental understanding). And, often students rebel when we try to show them why something works (our attempt to develop relational understanding). Skemp pointed to the difficulty of communicating using the term understanding if the sender is thinking of relational understanding and the receiver is thinking of instrumental understanding. This is especially relevant in the situation in which a teacher is teaching for relational understanding and a student is desiring instrumental understanding, or vice versa.

Skemp described some of the often-cited benefits of instrumental understanding, such as the rewards are more immediate and more apparent, and correct answers can be gotten more quickly. He then enumerated reasons for developing relational understanding. Relational understanding is more adaptable to new tasks and easier to remember. Relational knowledge is an effective goal in of itself because the need for rewards and punishments is greatly reduced. Relational schemas are organic in quality because they act as their own agents of change (Skemp 1987).

Today

In the 1980s, as psychologists and researchers in mathematics education were again studying understanding, the mathematics community was setting standards for what students should know and be able to do. One of the publications that helped set the stage for the need for standards was *Everybody Counts*. In this document, mathematics is described in a manner that called for understanding:

> Mathematics reveals hidden patterns that help us understand the world around us. . . .Mathematics is a science of pattern and order. . . .Its domain is . . . numbers, chance, form, algorithms, and change. . . . Mathematics relies on logic rather than on observation as its standard of truth, yet employs observation, simulation, and even experimentation as means of discovering truth. (National Research Council 1989, 31)

The *Standards* (National Council of Teachers of Mathematics 1989, 1991, 1995) emphasize the need for understanding in learning, teaching, and assessing mathematics. As important, the entire standards-based reform movement in schooling is described as one whose "focus [is] more on depth of understanding—how well students can reason with and use what they have learned—rather than on regurgitation of isolated facts." (McLaughlin, Shepard, and Day 1995, 9) Likewise, it is important that there is recognition beyond school mathematics for the need for understanding. As one collegiate mathematician recently stated, "large amounts of mathematics can be learned as sensible answers to sensible questions—that is, as part of mathematical sense making, rather than by 'mastery' of bits and pieces of knowledge" (Schoenfeld 1994, 59).

Why So Little Progress?

Tracing some of the writings of those who espoused understanding clearly points to the constant revisiting of this issue. Why, then, has learning with understanding not become part of our ethos?

I find the reasons why we should help students learn with understanding the most compelling part of the historical record. Even the brief quote of Colburn that opens this foreword causes me to pause and reflect on how and when I have gained understanding of mathematical ideas. I find the reasons of Brownell and Bruner for developing understanding to be consonant with my own learning experience and with evidence from my own teaching. They seem reasonable and sound; I am willing to make them part of my guiding principles. Yet, I still have many questions about my own teaching and the broader issues of reform.

Throughout the threads of the historical perspective you can feel the tension caused by the lack of clarity of what was meant by understanding. At times, advocates of understanding, such as Van Engen and Skemp, addressed directly this lack of clarity. At other times, there appeared to be an underlying assumption that we understood what we meant by understanding, meaning, or thinking. It is instructive to contrast the reasons given for developing understanding using an instrumental rather than a relational view of understanding. Although we have made progress on this issue, and this book will help us wrestle with what is meant by understanding, most of us have not completely resolved this issue. Pause for a minute and ask yourself what you mean by understanding. Then ask, how do you know when your students understand? What connections do you look for? What communication do you expect?

Has it only been this lack of definition or clarity of what is meant by understanding that has impeded our progress? I think not. There are many other influences that we need to consider to understand why we have lacked progress in the past and how we can make progress now.

Psychology

The prevalent psychological views during each of the periods were not always consonant with developing understanding. The contrasting views of Thorndike and Dewey—although each important to the later developing theories of psychology—may have prevented the moving forward of Dewey's view that is more closely aligned with developing understanding. Interpretations of behavioral psychology led us away from encouraging learning with understanding. Perhaps even more important, the prevalent psychology of learning was often not tied closely with a theory of teaching.

Mathematics

The prevailing view of school mathematics is one of rules and procedures, memorization and practice, and exactness in procedures and in answers. Many adults separate school mathematics, except for basic number ideas and skills, from the mathematics used in everyday life. There is no doubt that there are rules, need for practice, and exact answers. There is a need to store facts and procedures in memory. There are interesting mathematical problems that do not arise from a contextual or applied situation.

If mathematics is considered only as isolated facts and skills, then there is little use to encourage understanding. If mathematics is considered only as rules we memorize and practice, then thinking about these may be considered a waste of time. If mathematics is something we do only in school, then there is little point in developing a sense of when and how to apply mathematics. Through the years different slants on these views of mathematics have not lent themselves to encouraging a mathematics that makes sense.

Mathematical Learning

Our view of mathematical learning influences how we think about teaching. If we believe that education is mainly learning facts and procedures quickly and efficiently, if we believe that only certain students need or can learn mathematics, or if we believe that people are born with the ability to do or not to do mathematics, then our view will conflict with the development of understanding. If we look back in history, we see evidence for the first of these—educators during the time of Thorndike emphasized learning bits and pieces in an efficient manner. With regard to the second, it is only recently that we have moved from thinking of mathematics as an elite subject to one that is essential for all citizens. Finally, it is common in the United States for people to accept the ability argument, thus excusing many for not learning mathematics.

Educational Reform

Often we have had a view that educational problems can be fixed by changing one aspect, such as the curriculum, preparation of teachers, or assessment. For example, in the modern mathematics era, the focus mainly was on curriculum. Moving to classrooms that encourage understanding requires more than fiddling with one aspect, or adding more on to what is being done. It requires more substantive, long-term changes. It also requires a change in attitude and beliefs as well as in practice and expectations.

Another common view of education is that we need immediate results. We are an impatient nation and the payoff for teaching with understanding is often long-term. This requires articulation across levels as well as a change of expectations of what is learned and how it is learned. It depends upon a different view of accountability than we have now.

Teachers

I have saved to last what I consider the primary reason we have not made more progress. We often have failed to recognize the important contribution that teachers do and could make. Look at the history again. Colburn's description of what his book could do for teachers as well as Van Engen's and Wheat's acknowledgment of the teachers' role are quite different from what you will find in this book. Evident in this book is a position that respects and supports each teacher's knowledge, expertise, and beliefs. It is filled with the expectation that professionals, each with his or her own contributions, will work together as partners.

It is only when these views coincide that we will be able to make significant progress toward helping students develop understanding in mathematics. I believe that we are nearer to this confluence than in previous times, but the type of change called for in this book will not occur without involving the whole system and all the stakeholders of education. Just as changing one aspect of classroom practice will not suffice, changing only isolated classrooms will not be sufficient.

What Is This Book's Contribution to Developing Understanding?

In this book you will find guidance to begin or to renew your journey toward designing classrooms that encourage understanding. Each of us will find different ways to use this book as we think about understanding. Clearly, it influenced my thoughts about why we have not made more progress.

The view of understanding presented in Chapter 1 and illustrated throughout the book is deceptively simple. Actually it is broad and flexible, a definition that allows us to look at the many aspects of the proposed framework together and separately. The framework sets forth five dimensions: tasks, teacher's role, social culture, tools, and equity. The convergence of these dimensions along with the coherence of mathematical goals should help us move forward in building classrooms that provide all students the opportunity to build mathematical understandings that they can use throughout their lives.

Although I argued in the section about why we have not made progress that the developing of understanding cannot be done by adjusting only one dimension of your teaching, there are steps we can take to help us. Perhaps you will want to begin by looking at the nature of the mathematical tasks that you are presently using in your class. Do those tasks have the important characteristics of encouraging reflection and communication? Are there some ways that you can turn the tasks you are now using into ones that encourage understanding? What other aspects of mathematical tasks do you need to consider?

In this book you will find issues to discuss with colleagues to help you understand understanding. How does the social culture of your school affect the possibility of changing the culture of your classroom? Are there other mathematical tools needed in your classroom? How can you use mistakes as sites to encourage learning by everyone? How do you encourage students to share essential information? How do students determine the correctness of their mathematics? How do you involve each and every student in the sharing of their development of mathematical knowledge?

As you read and reflect on the examples of teaching, you will develop your own understanding of understanding. I found that one of the most meaningful paragraphs for me was in the story of Annie Keith's classroom. Describing that classroom (Chapter 7), the authors claim, "The constant is that students are always challenged to think and to try to make sense of what they are doing. They are challenged to take responsibility for monitoring their own learning and understanding. But learning is not an isolated individual activity; the students share ideas with one another and they learn from one another and learn to respect each other's ideas." As you read, find your own place where all the ideas seem to come together.

I firmly believe, as do the authors of this book, that the time for understanding has come and that it will become a part of our educational ethos. We have all learned from those that have espoused meaning or understanding in the past. This book adds to this discussion in a powerful way as it brings together many facets of the classroom—the tasks and tools, the teacher, the student, the environment, and the accessibility for and valuing of each and every student. It is a practical book that should help each and every one of us to reconsider what we are doing in our classrooms, whether they be at the grade levels described here, middle school, high school, college, or in our work with future and present teachers. No, you will not find all the answers for your own classroom, but you will find the important questions.

There are also positive signs that this is the time for understanding to be at the forefront because of the confluence of many of the forces that shape education. The information and changing world in which we live requires that we learn to learn, that we make the world understandable, and that we are confident of our abilities to do this. Mathematics in a technological world demands new skills and deeper understandings. The authors of this book use skill learning as a site for developing understanding, and build on the premise that this understanding allows for much deeper learnings.

The journey is not completed; we will continue to learn about teaching and learning with understanding. Along the way there will be milestones, one of which will be the ideas put forth in this book.

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Preface

This is a book about learning mathematics with understanding. What does it mean to design a classroom so that understanding is the primary goal? What would a system of instruction look like if we took seriously the goal of helping all students understand mathematics?

Our answers to these questions grew out of five years of discussions. Each of us are involved in ongoing projects that investigate the kinds of instruction that facilitate children's understanding of mathematics. At the invitation of the National Center for Research in Mathematical Sciences Education, we came together as a working group to examine the questions posed above. We shared our experiences and explored similarities and differences in the results of our projects and our interpretations of them. Although we shared the common goal of understanding what it means to teach for understanding, our individual projects are quite different and the similarities were not immediately apparent. Nevertheless, out of our continuing discussions grew a rather striking consensus about the features of classrooms that are essential for supporting students' understanding.

In this book we share our consensus about the essential features of classrooms for understanding mathematics. We also provide glimpses into our individual projects, and into the classrooms in which we have been spending time and from which we have drawn many of our ideas. By describing the essential features of classrooms that support students' mathematical understanding, and by providing pictures of several classrooms that exhibit these features, we hope to provide a framework within which teachers can reflect on their own practice, and think again about what it means to teach for understanding.

We wish to thank the many teachers and students who afforded us the opportunity of experiencing what it means to teach and learn mathematics with understanding. We also wish to thank the National Center for Research in Mathematical Sciences Education, University of

Wisconsin-Madison, and their granting agency, the Office of Educational Research and Improvement of the United States Department of Education (Grant No. R117G10002), for supporting our working group and making possible the preparation of this book. Of course, the opinions expressed in this book are ours and not necessarily those of the Office of Educational Research and Improvement.

1 Introducing the Critical Features of Classrooms

The world is changing. The societies that our students enter in the next decade and the next century will be different from those that we entered and different from those we see today. The workplace will be filled with new opportunities and new demands. Computers and new technologies are transforming the ways in which we do business, and future changes promise to be even more dramatic (Gates 1995). The skills needed for success will be different from those needed today. But the way in which societies will change, and the skills required of its citizens, are not fully predictable. Change is surely coming, but its exact nature is not entirely clear.

In order to take advantage of new opportunities and to meet the challenges of tomorrow, today's students need flexible approaches for defining and solving problems. They need problem-solving methods that can be adapted to new situations, and they need the know-how to develop new methods for new kinds of problems. Nowhere are such approaches more critical than in the mathematics classroom. Not only is technology making some conventional skills obsolete—such as high levels of speed and efficiency with paper-and-pencil calculations—it is also underscoring the importance of learning new and flexible ways of thinking mathematically.

All of this means that students must learn mathematics with understanding. Understanding is crucial because things learned with understanding can be used flexibly, adapted to new situations, and used to learn new things. Things learned with understanding are the most useful things to know in a changing and unpredictable world. There may be debate about what mathematical content is most important to teach. But there is growing consensus that whatever students learn, they should learn with understanding (National Council of Teachers of Mathematics [NCTM] 1989, 1991; Mathematical Sciences Education Board [MSEB] 1988).

Although important, usefulness is not the only reason to learn with understanding. If we want students to know what mathematics is, as a subject, they must understand it. Knowing mathematics, *really knowing it*, means understanding it. When we memorize rules for moving symbols around on paper we may be learning something, but we are not learning mathematics. When we memorize names and dates we are not learning history; when we memorize titles of books and authors we are not learning literature. Knowing a subject means getting inside it and seeing how things work, how things are related to each other, and why they work like they do.

Understanding is also important because it is one of the most intellectually satisfying experiences, and, on the other hand, not understanding is one of the most frustrating and ultimately defeating experiences. Students who are given opportunities to understand, from the beginning, and who work to develop understanding are likely to experience the kind of internal rewards that keep them engaged. Students who lack understanding and must resort to memorizing are likely to feel little sense of satisfaction and are likely to withdraw from learning. Many of us can recall instances from our own study of mathematics that resonate with these contrasting experiences. Understanding breeds confidence and engagement; not understanding leads to disillusionment and disengagement.

We begin, then, with the premise that understanding should be the most fundamental goal of mathematics instruction, the goal upon which all others depend. We believe that students' understanding is so important that it is worth rethinking how classrooms can be designed to support it. What kinds of classrooms facilitate mathematical understanding? That is the question this book is all about.

A Framework for Thinking About Classrooms

A primary thesis of this book is that classrooms that facilitate mathematical understanding share some core features, and that it is possible to tell whether classrooms support the development of understanding by looking for these features. In order to identify the features that support students' understanding, we need to set up a framework for analyzing classrooms. Our framework consists of five dimensions that work together to shape classrooms into particular kinds of learning environments: (a) the nature of the learning tasks, (b) the role of the teacher, (c) the social culture of the classroom, (d) the kind of mathematical tools that are available, and (e) the accessibility of mathematics for every student. We have found this framework useful because all classrooms can be analyzed along these five dimensions, regardless of the instructional approach. But more than that, the features that we believe are critical for facilitating understanding are found within these five dimensions. This means that the five dimensions form a framework both for examining whether a classroom is facilitating the development of understanding, and for guiding those who are trying to move their classrooms toward this goal. In other words, the framework can be used by teachers to reflect on their own practice, and to think about how their practice might change.

In this book we will look closely at each of these five dimensions. By presenting descriptions of each dimension and telling stories of classrooms that illustrate how the dimensions play out in real settings, we will identify what we think is essential for facilitating understanding and what is not. Some features within each dimension seem to be crucial for understanding, others seem to be optional. Through our explanations and illustrations, we will highlight the features that we believe are essential.

The book is organized into four parts. This introductory chapter provides an overview of what is to come. The chapter introduces many of the main ideas and raises questions that the reader might reflect on throughout the book. The second part consists of five brief chapters (Chapters 2–6) that describe the critical dimensions of classrooms designed for learning with understanding. Each chapter deals with one dimension and identifies and exemplifies those features that are essential for facilitating understanding. The third part illustrates how the critical features of classrooms can look in action. The four chapters (Chapters 7–10) each tell a story of a classroom. Although the classrooms may look different to a casual observer, we believe they share several core features within each dimension. The fourth part (Chapter 11) concludes the book by considering again the five dimensions, reviewing the critical features within each, and summarizing the ways in which these features can work together in classrooms.

Learning with Understanding

Most teachers would say that they want their students to understand mathematics, and in fact, that they teach for understanding. Teachers generally believe that understanding is a good thing. However, we have not always had a clear idea of what it means to learn mathematics with understanding, and we have had even less of an idea about how to tell whether a classroom was designed to facilitate understanding.

The reform efforts in mathematics education have, once again, directed the spotlight on understanding. Fortunately, we now are able to give a more complete description of what it means to learn with understanding and to teach for understanding. The reform documents themselves (NCTM 1989, 1991; MSEB 1988) provide some rich descriptions of what mathematical understanding looks like. The first four standards in NCTM's 1989 document highlight the importance of reasoning clearly, communicating effectively, drawing connections within mathematics and between mathematics and other fields, and solving real problems. All of these activities contribute to understanding and provide evidence for understanding.

Definition of Understanding

One of the reasons that it has been difficult to describe understanding in classrooms is that understanding is very complex. It is not something that you have or do not have. It is something that is always changing and growing. And understanding can be described from many different points of view. Because of its importance and complexity, there have been a number of recent descriptions of mathematical understanding, including those by Carpenter and Lehrer (1996), Davis (1992), Pirie and Kieren (1994), and Putnam et al. (1990). The reader may want to consult these and other sources for related but somewhat different descriptions of understanding.

A definition of understanding that works well for our purposes is one that has developed over many years and owes its existence to many psychologists and educators who have used and refined it in many contexts, including mathematics. This definition says that we understand something if we see how it is related or connected to other things we know (Brownell 1935; Hiebert and Carpenter 1992). For example, a teacher understands her student's anxiety about taking tests if she can relate the anxiety to other things she knows about the student, the current situation, and situations that the student may have encountered in the past. If she knows that the student has recently performed poorly on a major exam or that the student works very slowly and has trouble finishing tests on time, then she usually thinks she understands the student's anxiety a little better. The more relationships she can establish, the better she understands.

As another example, a student understands how to add 35 and 47 if she can relate this problem to other things she knows about addition and about the meaning of the numerals 35 and 47. Knowing that 35 is 3 tens and 5 ones and that 47 is 4 tens and 7 ones helps her understand several ways of combining the numbers. In both these cases, evidence for understanding is often provided in the form of explanations for why things are like they are, why the student is anxious, and why 35 and 47 is 82. Explanations are usually filled with connections, either implicit or explicit, between the target situation and other things that the person knows.

The definition of understanding in terms of relationships or connections works fine as a definition, but it does not reveal much about how people make connections. Furthermore, not all connections are equally useful. Some provide real insights and others are quite trivial. Some may even be inappropriate. To help think about how people make connections in mathematics and how they make connections that are useful, it is helpful to consider two processes that play an important role in the making of connections: reflection and communication.

Understanding Through Reflecting and Communicating

Two traditions in psychology have influenced our thinking about how students learn and understand mathematics—cognitive psychology with its emphasis on internal mental operations, and social cognition with its emphasis on the context of learning and social interaction (Hiebert 1992). The process of reflection is central for cognitive psychology, and the process of communication is central for social cognition. Although reflection and communication oversimplify these complex and influential traditions, they work well for our purposes because they provide important insights into how students construct understandings of mathematics and why the five dimensions of classrooms that we identified earlier are critical.

Reflection occurs when you consciously think about your experiences. It means turning ideas over in your head, thinking about things from different points of view, stepping back to look at things again, consciously thinking about what you are doing and why you are doing it. All of these activities have great potential for recognizing and building relationships between ideas or facts or procedures. In other words, stopping to think carefully about things, to reflect, is almost sure to result in establishing new relationships and checking old ones. It is almost sure to increase understanding.

Communication involves talking, listening, writing, demonstrating, watching, and so on. It means participating in social interaction, sharing thoughts with others and listening to others share their ideas. It is possible, of course, to communicate with oneself (reflection often involves such communication), but we will focus primarily on communication with others. By communicating we can think together about ideas and problems. This allows many people to contribute suggestions, so that we often can accomplish more than if we worked alone. Furthermore, communication allows us to challenge each other's ideas and ask for clarification and further explanation. This encourages us to think more deeply about our own ideas in order to describe them more clearly or to explain or justify them.

Communication works together with reflection to produce new relationships and connections. Students who reflect on what they do and communicate with others about it are in the best position to build useful connections in mathematics.

If it is true that reflection and communication foster the development of connections, then classrooms that facilitate understanding will be those in which students reflect on, and communicate about, mathematics. The question now becomes one of determining what kinds of classrooms encourage such activity. We believe that the five dimensions we identified earlier capture the aspects of classrooms that do just that. Before we explore these dimensions, we should deal with a common concern about understanding.

Is There a Trade-off Between Understanding and Skill?

Learning computational skills and developing conceptual understanding are frequently seen as competing objectives. If you emphasize understanding, then skills suffer. If you focus on developing skills, then understanding suffers. We believe that this analysis is wrong. It is not necessary to sacrifice skills for understanding, nor understanding for skills. In fact, they should develop together. In order to learn skills so they are remembered, can be applied when they are needed, and can be adjusted to solve new problems, they must be learned with understanding.

To some readers it may seem a bit ironic, but we have found that the learning of skills provides an ideal site for developing understanding. If students are asked to work out their own procedures for calculating answers to arithmetic problems and to share their procedures with others, they will necessarily be engaged in reflecting and communicating. Students who develop their own procedures for solving a problem, rather than imitating the procedure given in a textbook or demonstrated by the teacher, must reflect on the meaning of the numbers in the problem and on the operation involved in the calculation. Sharing their work involves more than just demonstrating a procedure; it requires describing, explaining, justifying, and so on as they are asked questions by their peers.

In spite of our belief that understanding and skills can and should develop together, we must make it clear that we assume the primary goal of mathematics instruction is conceptual understanding. But we must also make it clear that setting conceptual understanding as the primary goal does not mean ignoring computation skills. In fact, we have found that instruction for understanding can help students construct skills that can be recalled when needed, can be adjusted to fit new situations, and can be applied flexibly. In a word, we have found that such instruction can help students construct skills that they can actually use.

Dimensions and Core Features of Classrooms

Classroom instruction, of any kind, is a system. It is made up of many individual elements that work together to create an environment for learning. This means that instruction is much more than the sum total of all the individual elements. The elements interact with each other. It is difficult, if not impossible, to change one element in the system without altering the others. For example, suppose a teacher wanted to change the kinds of questions she asked. It is unlikely that she could change just the questions and leave everything else the same. Most likely, the nature of students' responses would change, the tasks for the students would change (at least the way students perceived the tasks), the ways in which the teacher listened and responded to students' responses would change, and so on. To repeat, instruction is a system, not a collection of individual elements, and the elements work together to create a particular kind of learning environment.

The dimensions we describe can be thought of as sets of features that are clustered around common themes. None of these dimensions, by itself, is responsible for creating a learning environment that facilitates students' constructions of understandings. Rather, they all work together to create such environments. Each of them is necessary, but not one, by itself, is sufficient.

The dimensions can also be thought of as a set of guidelines that teachers can use to move their instruction toward the goal of understanding. Just as students continually work toward richer understandings of mathematics, teachers continually work toward richer understandings of what it means to teach for understanding. The dimensions, and the core features within each dimension, provide guidelines and benchmarks that teachers can use as they reflect on their own practice.

The five dimensions will be described briefly here, and then elaborated in the following chapters. These introductions are intended to provide preliminary pictures of our classrooms. They herald the major issues that will appear throughout the book.

The Nature of Classroom Tasks

The nature of the tasks that students complete define for them the nature of the subject and contribute significantly to the nature of classroom life (Doyle 1983, 1988). The kinds of tasks that students are asked to perform set the foundation for the system of instruction that is created. Different kinds of tasks lead to different systems of instruction.

We believe that a system of instruction which affords students opportunities to reflect and communicate is built on tasks that are genuine problems for students. These are tasks for which students have no memorized rules, nor for which they perceive there is one right solution method. Rather, the tasks are viewed as opportunities to explore mathematics and come up with reasonable methods for solution.

Appropriate tasks have at least three features (Hiebert et al. 1996). First, the tasks make the subject *problematic* for students. We do not use this term to mean that students do not understand mathematics or that it is frustrating for them. Rather, problematic means that students see the task as an interesting problem. They see that there is something to find out, something to make sense of. Second, the tasks must connect with where students are. Students must be able to use the knowledge and skills they already have to begin developing a method for completing the task. Third, the tasks must engage students in thinking about important mathematics. That is, they must offer students the opportunity to reflect on important mathematical ideas, and to take something of mathematical value with them from the experience.

The Role of the Teacher

The role of the teacher is shaped by the goal of facilitating conceptual understanding. This means that the teacher sets tasks that are genuine mathematical problems for students so that they can reflect on and communicate about mathematics. Instead of acting as the main source of mathematical information and the evaluator of correctness, the teacher now has the role of selecting and posing appropriate sequences of problems as opportunities for learning, sharing information when it is essential for tackling problems, and facilitating the establishment of a classroom culture in which pupils work on novel problems individually and interactively, and discuss and reflect on their answers and methods. The teacher relies on the reflective and conversational problem-solving activities of the students to drive their learning.

This role of the teacher differs dramatically from the more traditional role in which the teacher feels responsible to tell students the important mathematical information, demonstrate the procedures, and then ask students to practice what they have seen and heard until they become proficient. Such a role fits with a system of instruction in which understanding is believed to come by listening carefully to what the teacher says. It does not fit a system in which understanding is constructed by students through solving problems.

The role we describe for the teacher does not exclude the teacher from participating in class discussions and sharing information with the students. The teacher is actively engaged in helping the students construct understandings. However, by intervening too much and too deeply, the teacher can easily cut off students' initiative and creativity, and can remove the problematic nature of the material. The balance between allowing students to pursue their own ways of thinking and providing important information that supports the development of significant mathematics is not an easy one to achieve (Ball 1993b; Dewey 1933; Lampert 1991). Indeed, it constitutes a central issue in defining the appropriate role of the teacher, an issue that will be revisited later.

The Social Culture of the Classroom

A classroom is a community of learners. Communities are defined, in part, by how people relate to and interact with each other. Establishing a community in which students build understandings of mathematics means establishing certain expectations and norms for how students interact with each other about mathematics. It must be remembered that interacting is not optional: it is essential, because, as we noted earlier, communication is necessary for building understandings. So, the question is not whether students should interact about mathematics, but how they should interact.

What kind of social culture fits with the system of instruction we are describing? What features are needed to create a social culture that would support the kinds of tasks and reinforce the role of the teacher that we have described? These are important questions because whether tasks are authentic problems for students, problems that allow and encourage reflection and communication, depends as much on the culture of the classroom as on the tasks themselves (Hiebert et al. 1996).

We can identify four features of the social culture that encourage students to treat tasks as real mathematical problems. The first is that ideas are the currency of the classroom. Ideas, expressed by any participant, have the potential to contribute to everyone's learning and consequently warrant respect and response. Ideas deserve to be appreciated and examined. Examining an idea thoughtfully is the surest sign of respect, both for the idea and its author. A second core feature of the social culture is the autonomy of students with respect to the methods used to solve problems. Students must respect the need for everyone to understand their own methods, and must recognize that there are often a variety of methods that will do the job. The freedom to explore alternative methods and to share their thinking with their peers leads to a third feature: an appreciation for mistakes as learning sites. Mistakes must be seen by the students and the teacher as places that afford opportunities to examine errors in reasoning, and thereby raise everyone's level of analysis. Mistakes are not to be covered up; they are to be used constructively. A final core feature of the social culture of classrooms is the recognition that the authority for reasonability and correctness lies in the logic and structure of the subject, rather than in the social status of the participants. The persuasiveness of an explanation or the correctness of a solution depends on the mathematical sense it makes, not on the popularity of the presenter. Recognition of this is a key toward creating a constructive community of learners.

Mathematical Tools as Learning Supports

A common impression is that the reform movement in mathematics instruction is mostly a recommendation to use physical materials to teach mathematics. We believe that the reform movement is about much more than using physical materials. We also believe that the discussion of mathematical tools would benefit from broadening the definition to include oral language, written notation, and any other tools with which students can think about mathematics.

Mathematical tools should be seen as supports for learning. But using tools as supports does not happen automatically. Students must construct meaning for them. This requires more than watching demonstrations; it requires working with tools over extended periods of time, trying them out, and watching what happens. Meaning does not reside in tools; it is constructed by students as they use tools.

In mathematics classrooms, just as in everyday activities, tools should be used to accomplish something. In the classrooms we are describing, this means that tools should be used to solve problems. Mathematical tools can help solve problems by functioning in a variety of ways. They can provide a convenient record of something already achieved (e.g., using written symbols to record the partial results while solving a multistep problem); they can be used to communicate more effectively (e.g., using square tiles to explain a method for finding the area of a surface); and they can be used as an aid for thinking (e.g., using base-ten blocks to see how 321 can be decomposed before subtracting 87).

Regardless of the particular tools that are used, they are likely to shape the way we think. Mathematical activity requires the use of tools, and the tools we use influence the way we think about the activity. Another way to say this is that tools are an essential resource and support for building mathematical understanding, and the tools students use influence the *kinds* of understandings they develop (Fuson et al. 1992). Remember that understanding is a complicated thing. It is not all or nothing. It is made up of many connections or relationships. Some tools help students make certain connections; other tools encourage different connections.

An example can be drawn from second-grade arithmetic. When students are first learning to add and subtract numbers with two or more

digits, there are many tools they might use. These include base-ten blocks, connecting cubes, hundreds charts, flip cards, written numbers, as well as language skills such as counting by tens and using a special vocabulary that highlights the tens and ones groupings (e.g., 18 is 1 ten and 8 ones). It is possible to imagine any of these tools being used in classrooms that incorporate the core features mentioned earlier: classrooms in which students encounter genuine problems; where the teacher encourages students to work out and share their own strategies; and where students respect each other's ideas. In other words, it is possible to imagine students using any of these tools to construct understandings. But it is also reasonable to believe that different tools may encourage different understandings. Students who use base-ten blocks may tend to develop different strategies (and consequently learn somewhat different things about numbers) than students who build on well-developed counting skills (this is a complex issue and will be examined further in Chapter 5). It should be noted that some of the variability apparent in the stories of classrooms (Chapters 7–10) is due to different choices of tools.

Equity and Accessibility

We believe that every student has the right to understand what they do in mathematics. Every student has the right to reflect on, and communicate about, mathematics. Understanding is not just the privilege of the high-achieving group. This is not a blue-sky belief that is out of touch with reality. Our experience is that, given classrooms like those we describe here, girls and boys at all levels of achievement and from all backgrounds can understand what they do in mathematics. More than that, understanding supports improved performance for students at all levels (Carey et al. 1993; Hiebert and Wearne 1993; Hiebert et al. 1991). That is, understanding is just as important for low achievers as for high achievers if we hope to raise levels of achievement above those in the past.

Equitable opportunities for all students sit squarely on the core features of classrooms described to this point. Tasks of the kind described in Chapter 2 must be accessible, at some level, to all students. The role of the teacher (Chapter 3) and the social culture of the classroom (Chapter 4) both point to the necessity of listening carefully to what each student says with a genuine interest in the ideas expressed (Paley 1986). Listening in this way does two things: It conveys a fundamental respect for the student, and it allows the teacher and peers to know the student as an individual. Both of these remove stereotypes and eliminate expectations that might be tied to particular group memberships. Equity, in part, means that each student is treated as an individual, and listening, *really listening*, is one of the best ways to encourage such treatment.

Equity contributes to the other dimensions as well as being a natural consequence of them. Establishing an appropriate social culture, for example, depends on every student participating as a member of the mathematics community. Learning opportunities arise as different ideas and points of view are expressed. To the extent that some students do not participate in the community, the learning opportunities are constrained. A rich, fully functioning community requires everyone's participation.

It is important to note that the notion of equity, as we interpret it, is not an add-on or an optional dimension. It is an integral part of a system of instruction that sets students' understanding of mathematics as the goal. Without equity, the other dimensions are restricted and the system does not function well. All five dimensions and the critical features within each are needed for the system to work.

Figure 1–1 provides a summary of the five dimensions and the features within each that we think are essential. Readers might wish to refer to the figure as a reminder of the major points in this chapter, and as an advance organizer for Chapters 2–6. These chapters will describe the core features in more detail and also will identify some optional features within the dimensions.

DIMENSIONS	CORE FEATURES
Nature of Classroom Tasks	Make mathematics problematic Connect with where students are Leave behind something of mathematical value
Role of the Teacher	Select tasks with goals in mind Share essential information Establish classroom culture
Social Culture of the Classroom	Ideas and methods are valued Students choose and share their methods Mistakes are learning sites for everyone Correctness resides in mathematical argument
Mathematical Tools as Learning Supports	Meaning for tools must be constructed by each user Used with purpose--to solve problems Used for recording, communicating, and thinking
Equity and Accessibility	Tasks are accessible to all students Every student is heard Every student contributes

1–1 Summary of dimensions and core features of classrooms that promote understanding

History of This Project

This book is an outgrowth of the collaboration of researchers from four research and development projects. During the past five years we met regularly to discuss our projects and examine differences and similarities in our approaches. Out of our discussions emerged a gradual but growing consensus about the essential features of classrooms that are designed to support students' understanding. This book describes our consensus. It represents our best collective thinking about these issues, thinking that is informed by evidence we have collected, observations of many different kinds of classrooms, discussions with many different teachers, and our reflections and communications with each other.

All of our projects focus on arithmetic in elementary school, with special attention to students' initial learning of multidigit addition and subtraction. This means that most of our examples will be taken from these topics and that the classroom stories presented later will describe lessons that involve whole number arithmetic. Although we recognize that other mathematics topics present some unique, specific questions, we believe that many of the issues we address and observations we provide are appropriate for mathematics teaching and learning in general. We pitched our descriptions at a level that could be applied to the teaching of any mathematical topic. For example, the five dimensions we identified and the core features within those dimensions are equally applicable to a range of topics and ages of students. Readers who would like to apply the ideas to, say, the teaching of percent in seventh grade, might need to build a few bridges on their own, but we believe that the crucial ideas are sufficiently alike that such constructions are possible.

The four projects were all conceived with an eye toward increasing students' understanding. Evidence of attention to the five dimensions of classrooms are apparent in each project, but with different configurations and different emphases. In order to provide a sense of the roots of our collective thinking, it is useful to provide a brief glimpse into the nature of the projects.

The four projects, in alphabetical order, are Cognitively Guided Instruction (CGI) directed by Thomas Carpenter, Elizabeth Fennema, and Megan Franke at the University of Wisconsin–Madison; Conceptually Based Instruction (CBI) directed by James Hiebert and Diana Wearne at the University of Delaware; Problem Centered Learning (PCL) directed by Piet Human, Hanlie Murray, and Alwyn Olivier at the University of Stellenbosch in South Africa; and Supporting Ten-Structured Thinking (STST) directed by Karen Fuson at Northwestern University.

All of the projects study learning and teaching in elementary classrooms, but they do so in somewhat different ways. CGI does not develop curricula nor design instruction. The primary goal is to help teachers acquire knowledge of children's mathematical thinking and then to study how teachers use their knowledge to design and implement instruction. CBI and STST design new instruction, work with teachers to implement it, and study the nature of students' learning in these classrooms. PCL is a large curriculum development and teacher training project. Teaching and learning are studied as teachers implement the PCL approach.

Despite the different orientations of the projects, the classrooms involved in each project show some striking similarities. The learning of basic number concepts and skills is viewed as a problem-solving activity rather than as the transmission of rules and procedures. Teachers allow students the time needed to develop their own procedures and do not expect all students to use the same ones. Class discussions involve sharing alternative methods and examining why they work. Teachers play an active role by posing problems, coordinating discussions, and joining students in asking questions and suggesting alternatives. In short, it appears that classrooms across the four projects employ the system of instruction we will describe, and exhibit the core features shown in Figure 1–1.

Differences also exist, not only among classrooms in different projects, but among classrooms within the same project. The differences arise from differences in the tasks selected, the kind of information provided, and the tools used to solve problems. For example, in some of the STST studies and in the CBI classrooms, students work with base-ten blocks and are helped to build connections between the blocks and written numerals, and between joining and separating actions on the blocks and adding and subtracting with numerals. In contrast, students in PCL classrooms do not use base-ten blocks and do not spend time building connections between structured manipulative materials and written numerals. Rather, they initially engage in a variety of counting activities and then develop arithmetic procedures from these understandings.

The contrast between these classrooms and the differences that would be immediately apparent to a casual observer highlight one of our central messages: Classrooms that promote understanding can look very different on the surface and still share the core features we have identified. Designing classrooms for understanding does not mean conforming to a single, highly prescribed method of teaching. Rather, it means taking ownership of a system of instruction, and then fleshing out its core features in a way that makes sense for a particular teacher in a particular setting. Chapters 7–10 illustrate further the ways in which classrooms can look different and still be very much the same.

Summary

Out of our four projects has emerged a consensus about what it means to understand mathematics and what is essential for facilitating students' understanding. We agree on the following principles: First, understanding can be characterized by the kinds of relationships or connections that have been constructed between ideas, facts, procedures, and so on. Second, there are two cognitive processes that are key in students' efforts to understand mathematics—reflection and communication. Third, there are five dimensions that play a prominent role in defining classrooms in terms of the kinds of learning that they afford: the nature of the tasks students are asked to complete, the role of the teacher, the social culture of the classroom, the mathematical tools that are available, and the extent to which all students can participate fully in the mathematics community of the classroom. Fourth, there are core features within these dimensions that afford students the opportunity to reflect on and communicate about mathematics, to construct mathematical understandings.

In the remainder of the book, we address these issues in more detail and provide extensive illustrations of how classrooms might look. Although we draw on our immediate experience working with primary-grade students on multidigit addition and subtraction and present many examples from this work, there are general principles here that could be applied to other age groups and other mathematical topics. We trust that we have shaped our descriptions and discussions so that such applications are possible for the reader to make.