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Designing a High School Mathematics Curriculum for All Students

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The Interactive Mathematics Program has created a four-year secondary-school curriculum designed to provide a sound mathematics background for all secondary students. This article describes various characteristics that must go into such a curriculum in order for it to be successful with all students and discusses crucial elements that are needed for its successful implementation.

Many mathematics educators agree that in order to develop "mathematical power" in our students, the primary focus of mathematics education must shift from the learning of procedures to the solving of complex problems. The goal in this shift is not simply to develop mathematical power in a few students, but to develop it in all students.[1]

The need for such a shift in educational direction has been prompted in part by changing needs in the workforce. In 1989, the National Research Council published *Everybody Counts,* a report on the future of mathematics education. That report contended that "for lack of mathematical power, many of today's students are not prepared for tomorrow's jobs. In fact, many are not even prepared for today's jobs" (National Research Council 1989, p. 1). The report continued: "From the accountant who explores the consequences of changes in tax law to the engineer

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who designs a new aircraft, the practitioner of mathematics in the computer age is more likely to solve equations by computer-generated graphs and calculations than by manual algebraic manipulations. Mathematics today involves far more than calculation; clarification of the problem, deduction of consequences, formulation of alternatives, and development of appropriate tools are as much a part of the modern mathematician's craft as are solving equations or providing answers" (National Research Council 1989, p. 5).

The National Research Council recommended that the United States adopt as a national goal the development of "new curricula appropriate to the mathematical needs of the twenty-first century" (National Research Council 1989, p. 88).

Also in 1989, the National Council of Teachers of Mathematics (NCTM) set forth new goals for mathematics education, which are based on the changing needs of society. To meet these goals, NCTM's *Curriculum and Evaluation Standards for School Mathematics* (subsequently referred to here as the NCTM *Standards*) called for a new curriculum that would de-emphasize paper-and-pencil skills, integrate algebra and geometry, expand the curriculum to include probability, statistics, and discrete mathematics, fully utilize computers and graphing calculators, and contain numerous small group and individual investigations.

The NCTM *Standards* also emphasized that the new curriculum must be available for all students:

> The social injustices of past schooling practices can no longer be tolerated. Current statistics indicate that those who study advanced mathematics are most often white males. Women and most minorities study less mathematics and are seriously underrepresented in careers using science and technology. Creating a just society in

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> which women and various ethnic groups enjoy equal opportunities and equitable treatment is no longer an issue. Mathematics has become a critical filter for employment and full participation in our society. We cannot afford to have the majority of our population mathematically illiterate. Equity has become an economic necessity (National Council of Teachers of Mathematics 1989, p. 4).

The challenge of meeting this call for equity—for inclusion of all of our students in the study of mathematics—is the main focus of this article.

The Interactive Mathematics Program (IMP)[2] has developed a new four-year curriculum that incorporates the recommendations of the NCTM *Standards* and that is intended to provide mathematical power for all secondary students who use it.[3] This article describes various characteristics that must go into such a curriculum and crucial elements that are needed for its implementation.

We began our work with the belief that a secondary mathematics curriculum could be developed that would teach needed skills as well as develop mathematical power, and that it would still be successful with the full range of students. It was also our belief that a single curriculum, used in heterogeneous classes, was the most effective way to provide equal access. That is, we rejected the idea of "separate but equal" curricula.[4]

This article discusses the principles that went into the design and development of the curriculum. These principles have been implicit in many projects that have worked to provide more students with successful experiences in mathematics.[5] As we implemented these principles, we had to make many decisions, some on the basis of mathematics education research literature and some on intuition. Since each of the four of us either taught the curriculum or observed classes regularly during the development process, all decisions were guided by classroom experience as well. Each component of the curriculum went through at least three preliminary drafts, and each draft was field tested in classrooms with diverse student populations.

Principles of Curriculum Development

The following are four of the principles that guided our curriculum development:[6]

1. Students must feel at home in the curriculum.
2. Students must feel personally validated as they learn.

3. Students must be actively involved in their learning.
4. Students need a reason for doing problems.

We clarify each of these principles in turn and discuss what they mean in terms of the curriculum.

Students Must Feel at Home in the Curriculum

Mathematics teachers know firsthand that their discipline has a special place in the mind of the public. Telling a new acquaintance that one teaches mathematics elicits such comments as, "I never could do mathematics" or "You must be smart." The latter remark is often delivered with a look that says, "You must be strange."[7] By the time students reach high school, many of them have the attitude that mathematics is for special people and that not everyone can learn mathematics. They walk into a mathematics classroom feeling "It's not for me," and even those who are good at mathematics often feel a bit embarrassed by this fact.[8] For a mathematics curriculum to succeed with all students, it must take these perceptions of the discipline into account and be designed to change them.

In the past, many programs trying to promote mathematical equity have sought to help the members of the excluded groups match the models of success.[9] For instance, they included workshops on "how to think mathematically," which meant "how to think the way the mathematical establishment thinks." Such programs expected students to adapt to the curriculum and to the existing style of instruction.

Our approach has been different. Rather than mold students to fit the curriculum, we have tried to create a curriculum that would fit a wider range of students. That is, in order to achieve our goal of mathematical inclusion, we tried to develop the IMP curriculum so it would be consistent with the needs and tastes of all students. This has meant accommodating a wider range of learning styles and a wider range of cultures than traditional curricula have done.

Introducing ideas concretely.—There is evidence (Turkle and Papert 1992) that many people prefer to use more concrete approaches than those used in traditional mathematics textbooks. They prefer to proceed by getting their hands dirty in the details of a problem instead of by working abstractly. Those who proceed in such a style, staying close to the initial setting, are often good problem solvers. The traditional insistence on a more abstract approach can discourage these people, by suggesting to them that their natural strategies are inappropriate and that mathematics is a subject they cannot master.

For example, there is an important category of mathematical problems involving counting the number of ways that objects can be placed in sequences or put into groups. Traditional texts often present these "permutation and combination" problems by introducing the abstract idea of combinatorial coefficients. But many people are uncomfortable with this as an initial approach and prefer to begin by explicitly listing the different ways and reasoning directly in terms of the situation rather than from abstract principles.

In the IMP curriculum, we try to introduce new ideas as concretely as possible, even if the ideas are ones that students may have seen before. For example, most high school students know something about angles, but as college faculty can attest, students' knowledge about angles is often quite vague, and many students lack an intuitive understanding of what angles are about. When the IMP curriculum first discusses angles, we introduce them as turns, and students turn their bodies specific amounts so they can actually "feel" the angles. Similarly, area is introduced through geoboards, with physical representations (small squares on the board) as units: students count the number of these squares inside areas that are surrounded by rubber bands.

One danger of these approaches is that students who do understand angles and area measurement will feel patronized and think that the work is not worthwhile because they already know the mathematics. To deal with this issue, we have incorporated challenging problems within the concrete work.

For example, the activity shown in figure 1 is used when geoboards and measurement of area are introduced. The first problem simply asks students to calculate areas concretely. This can be a challenging task for many students. For students who finish more quickly, parts 2 and 3 generally provide ample challenge and are more interesting.

Representation in the curriculum.—A great quantity of anecdotal evidence shows that many young women and members of minority groups feel alienated in mathematics classes.[10] In fact, excelling in mathematics sometimes makes members of these groups feel less a part of their female or minority community.

One response many curriculum writers have made to this problem has been to include pictures and names of women and minorities as characters in textbook problems. The Interactive Mathematics Program problems do include roughly equal numbers of males and females and often use characters' names that are identified with minority groups. We also include photographs of field-test classrooms that show a diverse student population. Unfortunately, some students perceive the IMP curriculum as having an overwhelming majority of female characters, perhaps because these students are used to seeing a preponderance of male names in their texts or because they perceive male names as gender neutral. Similarly, when Asian, Hispanic, or Arabic names are included for characters, some students wonder why the characters have such "funny" names.

The issue of how to be inclusive of women and minorities without being perceived as strange by males or whites presents a dilemma for curriculum writers addressing a national audience. The use of female names for half of the characters does make some male readers feel less at home. In parts of the country that are not racially or ethnically diverse, the inclusion of unfamiliar names can make the curriculum seem foreign. Yet the inclusion of such names draws in other students. The Interactive Mathematics Program's decision has been to include names and characters that are representative of the country as a whole, reasoning that it is better to have some role models for each student even if the character names may seem strange to other students.

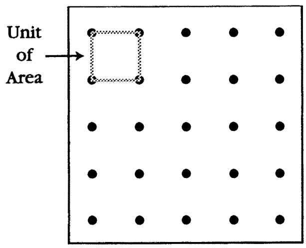
As curriculum developers, we were confronted with a similar dilemma in deciding what contexts to use for the mathematics. Can a context be equally interesting to all students? The Interactive Mathematics Program has made the decision to seek situations that will be interesting to most students in a classroom, trying to ensure that there are some situations in the curriculum that are particularly interesting for each student.

As with character names, this approach also creates difficulties. The Interactive Mathematics Program has been criticized, for example, for using a baseball situation for the central problem of one unit (Fellows et al. 1994, p. 7). The objection was that boys are generally more interested in baseball than girls are. But most genuinely challenging situations appeal more to one group than to another. The alternative is to limit the contexts for the mathematics to bland situations that really interest no one. Rather, we prefer to include a variety of situations, some of which might appeal more to one group than to another, and we have tried to strike a balance so that every student is intrigued by many of the contexts. (Of course, we have also sought to avoid any contexts that would be genuinely offensive to anyone.)

The social environment of the classroom.—Another way to make many students feel more at home is to create a classroom environment with more student interaction, in which students can speak informally with each other. In IMP, as in many reform classrooms, students spend much of their time working in small groups. This approach is very appealing to students who like to interact with others. Such a collaborative approach appears to make mathematics more appealing to female students, yet we also have not seen any indication that this approach alienates male students.

It should be noted that the reasons for using groups in mathematics

Nailing Down Area



In this activity, the unit of area will be the smallest square on the geoboard, such as the one shown here.

1. Construct each of figures A through L on your geoboard and find their areas. Record your results.

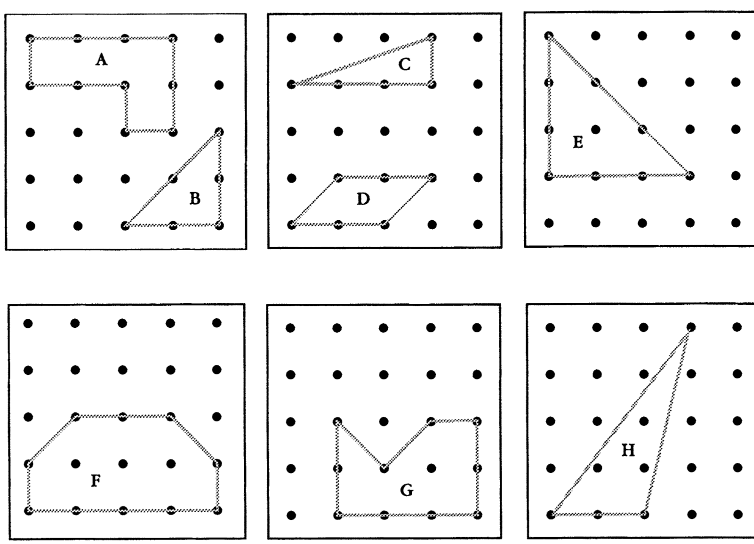


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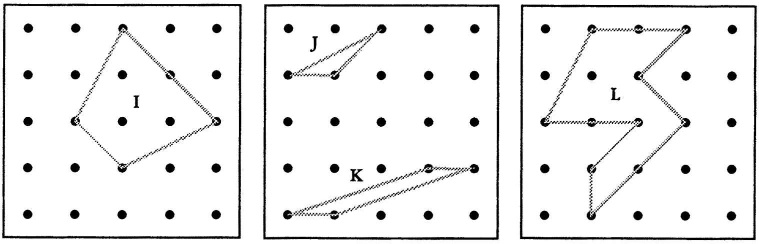
FIG. 1

Do Bees Build It Best? Classwork



2. Create triangles on the geoboard whose area is half a unit. Find as many different-shaped triangles with this area as you can and record your results on geoboard paper.

3. Create quadrilaterals (four-sided figures) on the geoboard whose area is exactly 1 unit. Find as many different-shaped quadrilaterals with this area as you can and record your results on geoboard paper.

Adapted from *About Teaching Mathematics* by Marilyn Burns, page 99. © 1992 Math Solutions Publications. Used by permission.

FIG. 1 (*continued*)

classes go beyond the desire to make students feel more at home. For example, students working in this way deepen their understanding by having to explain their ideas and are exposed to some cognitive dissonance.[11] The ability to work cooperatively is also highly valued in the workforce.[12]

Involving the community.—A final way IMP has worked to make students feel more at home in mathematics classes is to involve their parents, and thus their communities, in mathematics. Students are asked periodically to explain an idea to someone outside their classroom, as illustrated in figure 2.

Another opportunity for students to involve their families in mathematics is provided by "Problems of the Week" (POWs), which are complex problems that students are given at least a week to complete. Often these problems are engaging enough that the whole family gets involved in thinking about them, sometimes at the dinner table. Figure 3 shows an example. The concrete nature of this task makes it accessible to people of varied mathematical backgrounds, so it is well suited to the family setting. (Although students are encouraged to work with others on such problems, they are expected to develop their own write-ups of their work and to acknowledge the contributions of others.)

A broader way of involving students' communities in their mathematics classes is to have family IMP nights. At such events, parents, students,

Homework 8 Negative Reflections

When you first learned about exponents, their use was defined in terms of repeated multiplication. For example, you defined 2^5 as $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$.

With that repeated-multiplication definition, the exponent had to be a positive whole number. Now you have seen a way to make new definitions that allow zero and negative integers to be exponents.

1. Write a clear explanation summarizing what you have learned about defining expressions that have zero or a negative integer as an exponent. Then explain, using examples, why these definitions make sense. Give as many different reasons as you can, and indicate which explanation makes the most sense to you.
2. Show your explanation to an adult, and ask that person whether it made sense to him or her. Then write about the person's reaction.

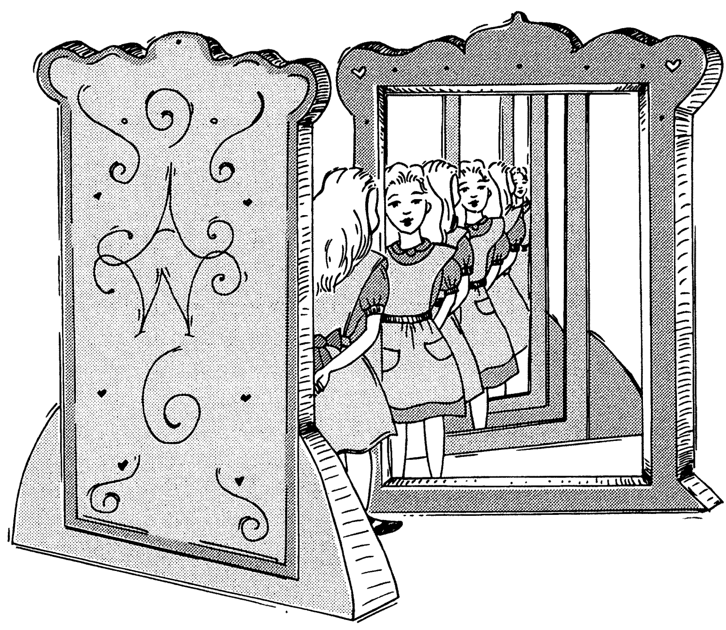


FIG. 2

POW 13 Corey Camel

Consider the case of Corey Camel—the enterprising but eccentric owner of a small banana grove in a remote desert oasis.

Corey's harvest, which is worth its weight in gold, consists of 3000 bananas. The marketplace where the harvest can be sold is 1000 miles away. However, Corey must walk to the market, and she can carry at most 1000 bananas at a time. Furthermore, being a camel, Corey eats one banana during each and every mile she walks (so Corey can never walk anywhere without bananas).

The question is this:

How many bananas can Corey get to the market?

Write-up

1. *Problem statement*
2. *Process:* You will also work on a mini-POW that relates to this POW. In discussing your process on this POW, note how your work on that mini-POW helped you. Also, be sure to discuss all of the methods you tried in order to solve the POW itself.
3. *Solution:*
   a. State your solution (or solutions) as clearly as you can.
   b. Do you think your solution is the best possible one? Explain.
   c. Explain how and why the answer to this POW is related to the answer to the mini-POW.
4. *Evaluation*

FIG. 3

friends, and relatives work together in groups on engaging problems. Generally, people from one family are mixed in groups with people from other families. Almost all schools with the IMP program hold such events and have found them very beneficial for student learning.

Attracting families to such events may be difficult, since many adults feel estranged from schools and especially from mathematics classes. One way to promote attendance is to have students run the event and play prominent roles. Parents then come just as they would if their child were in a school play. Other useful strategies include advertising that graphing-calculator technology will be used at the event, offering students extra-credit points for each person from the community they bring, having one enthusiastic parent call others, providing child care, and making the IMP evening a potluck event.

Schools have found that having successful family IMP nights does make a difference. Teachers report that many students become more involved in mathematics class and turn in more homework after such events.

Students Must Feel Personally Validated as They Learn

Most texts introduce algorithms for tasks such as solving systems of equations by giving students a step-by-step procedure. Curricula using this method are trying to show the student exactly what to do in order to be successful. But such methods do not foster understanding, and if students forget the procedure, they may be left with a feeling of hopelessness. *Everybody Counts* describes the situation: "Unfortunately, as children become socialized by school and society, they begin to view mathematics as a rigid system of externally dictated rules. . . . Their view of mathematics shifts gradually from enthusiasm to apprehension, from confidence to fear. Eventually, most students leave mathematics, convinced that only geniuses can learn it. Later, as parents, they pass this conviction on to their children. Some even become teachers and convey this attitude to their students" (National Research Council 1989, p. 44).

The NCTM *Standards* offers a more positive vision of mathematics instruction: "A major goal of mathematics instruction is to help children develop the belief that they have the power to do mathematics and that they have control over their own success or failure. This autonomy develops as children gain confidence in their ability to reason and justify their thinking. It grows as children learn that mathematics is not simply memorizing rules and procedures but that mathematics makes sense, is logical, and is enjoyable" (National Council of Teachers of Mathematics 1989, p. 29).

To implement this vision, IMP assignments actively encourage students to make sense of the mathematics they are learning. As students work on a task, we give them the freedom to solve the problem in their way—not in some predetermined way. Encouragement and praise for students' unique methods of solution is an important teacher task. Asking for more than one way to solve a problem informs the students that we value their ways of doing problems. Seeing that there are many approaches to a problem gives students more options and frees them up to explore future problems without worrying about remembering the "right" approach.

The Interactive Mathematics Program asks teachers to use heterogeneous groups, randomly assigning students to new groups every two weeks or so. We find that over time students learn to respect different strengths in each other. Some students see the abstract path, others have a novel, visual approach, and some ask the right questions. It is an important function of the teacher to make sure that one or two students do not do all of the group's work, but that all students are actively involved in the task at hand.

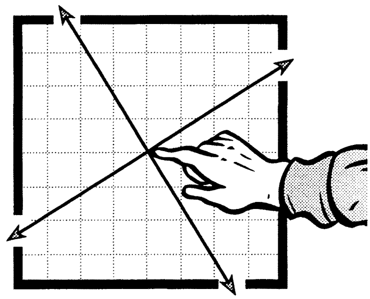
Many of the reform projects, including IMP, ask students to discover mathematical principles on their own, rather than simply tell them the answer. Figure 4 shows an IMP assignment in which students develop a method for solving systems of equations in two variables. Having students work on such problems in groups allows them to pool their strategies and develop their own procedures.

But groups do sometimes get stuck and reach a point where no one knows what to do next. If a teacher leaves this situation alone, students may get inspired, but they may also get frustrated or bored. One job of the teacher is to know when and how to step in. To prepare teachers to deal with situations like this, a curriculum program must include a professional development program, giving teachers guidance about the degree and kind of support for students that is appropriate. The Interactive Mathematics Program teacher guides provide extensive pedagogical strategies, including sample questions to ask students that will start them thinking along some productive path.

In the case of the activity in figure 4, a teacher might ask, "What is true of all points on the line $y = 3x$?" or "How would you get the y-coordinate from the x-coordinate for a point on this line?" The goal is to elicit an answer to the effect that the y-coordinate is three times the x-coordinate for all points on the line. The teacher can then ask a similar question for the line $y = 2x + 5$ and help students connect the two equations by asking what happens at the point of intersection.

Not all groups will need questions or hints. One of the advantages of having students work in small groups is that teachers can individualize

Get the Point



In solving problems like the cookie problem, it is helpful to know how to find the coordinates of the point where two lines intersect. As you have seen, this is equivalent to finding the common solution to a system of two linear equations with two variables.

You have probably done this already using either guess-and-check or graphing. Your goal in this activity is to develop an algebraic method, by working with the equations of the two straight lines.

Your written report on this activity should include two things.

- Solutions to Questions 1a through 1e
- The written directions your group develops for Question 2

1. For each of these pairs of equations, find the point of intersection of their graphs by a method other than graphing or guess-and-check. When you think you have each solution, check it by graphing or by substituting the values into the pair of equations.

   a. $y = 3x$ and $y = 2x + 5$

   b. $y = 4x + 5$ and $y = 3x - 7$

   c. $2x + 3y = 13$ and $y = 4x + 1$

   d. $7x - 3y = 31$ and $y - 5 = 3x$

   e. $4x - 3y = -2$ and $2y + 3 = 3x$

2. As a group, develop and write down general directions for finding the coordinates of the point of intersection of two equations for straight lines using an algebraic method, without guessing or graphing. In developing these instructions, you may want to make up some more examples like those in Question 1, either to get ideas or to test whether your instructions work.

   Make your instructions easy to follow so someone else could use them to "get the point."

FIG. 4

the input so each group of students is challenged but not too frustrated. Personal attention from the teacher and the process of developing their own ideas both provide validation to students. Of course, having students work alone would allow the teacher to individualize even more, but this deprives students of the cognitive and affective stimulation of their peers, both of which are important components of their development.

Creating an atmosphere of respect.—Another important task for the teacher in validating students is to create a classroom atmosphere of respect for individuals. Such an atmosphere must begin with the teacher, for the teacher needs to believe that every student is capable of achieving mathematical power. In IMP, we do come across teachers who are uncomfortable with having students work on open problems and feel the need to put additional structure on the activity. Watching such a teacher, one gets the impression that the teacher does not trust the students to learn on their own. The teacher seems to be thinking, "If I don't tell them, they won't learn." As a result, the students do not feel the same confidence in their mathematical ability as others working in the same school. This is another area in which a new curriculum must be accompanied by a professional-development program for teachers. We have seen tremendous growth among IMP teachers as they see positive results from allowing their students' ideas to determine the direction for the class.

Beyond believing in their own students, teachers need to ensure that students come to believe in each other. Students need to learn to listen when their peers speak and to be respectful yet critical. The IMP curriculum provides many opportunities for this, since students are constantly presenting their ideas both in small groups and to the whole class. By continually attending to students' attitudes toward each other and by modeling respect, teachers can create a healthy classroom atmosphere.

One important aspect of creating an atmosphere of respect involves the handling of student errors. It is an important validation of students to acknowledge that mistakes are inevitable and that they provide a meaningful way to learn. Once students feel comfortable making mistakes, they do not have to hide their initial thinking on a problem or be embarrassed if they are wrong. Although most of the work in this direction must be done by the teacher, curriculum can help develop this comfort level. One technique that IMP uses is to have students describe their processes in solving problems of the week. They are expected to discuss what they tried, even if it led them nowhere.

Both the teacher and the curriculum must avoid making students feel inadequate for not understanding something immediately. Many mathematics texts, especially at the college level, use phrases such as "It is clearly true" and "We easily see," which become put-downs to any reader for whom it is not clear or who does not easily see. To validate students, a program must be careful to avoid assuming that what is clear to the writer will be clear to the reader.

Grades and validation.—Grading is a difficult issue in classrooms where teachers are concerned about enhancing students' self-images and is of particular concern in trying to provide encouragement to individuals who have previously not had success. Although teachers need to have clear standards for all their students, they must also take into account that some students may have mathematics backgrounds that are not very firm or that some students may be overcoming language barriers.

The topic of grades raises many questions. What is their purpose? Are they a form of judgment or a motivation for students? (Grades certainly have both uses in society.) Is a grade supposed to measure a student's knowledge or measure the student's growth in mathematics? Is it fair to judge all students by the same standard? These are questions that every teacher struggles with, especially those who are teaching in a heterogeneous setting.

In IMP, as in all programs, grading is the teacher's responsibility. Rather than give teachers guidelines for assigning grades, we use in-service workshops to provide teachers with a forum for discussing issues around grading.

Students Must Be Actively Involved in Their Learning

The Interactive Mathematics Program makes use of the constructivist theory of learning.[13] Thus, IMP students are more actively involved in the learning of mathematics than students in traditional classrooms. The rationale for an active classroom is explained in the NCTM *Standards:*

> In many classrooms, learning is conceived of as a process in which students passively absorb information, storing it in easily retrievable fragments as a result of repeated practice and reinforcement. Research findings from psychology indicate that learning does not occur by passive absorption alone. . . . Instead, in many situations individuals approach a new task with prior knowledge, assimilate new information, and construct their own meanings. . . . This constructive, active view of the learning process must be reflected in the way much of mathematics is taught. . . . Our ideas about problem situations and learning are reflected in the verbs we use to describe student actions (e.g., to investigate, to formulate, to find, to verify) throughout the *Standards* (National Council of Teachers of Mathematics 1989, p. 10).

Beyond the reasons based on formal research for maintaining an active classroom, classroom observations and teacher reports indicate that active classrooms maintain the interest of many students who do not do well in traditional, passive situations. Often, maintaining an active classroom means replacing a teacher-led, whole-class discussion with a small-group activity that provides more immediate engagement for students.

Adapting curriculum for active involvement.—In an early version of our curriculum, after students discovered the Pythagorean theorem in a small-group activity, the teacher led them through a proof of the theorem as a whole class. Teachers reported that, while the initial discovery went well, many students then "tuned out" the whole-class discussion of the proof. The students who tuned out were not disruptive, but it was clear that their minds were elsewhere. To deal with this problem, and to involve all of the students actively in the proof, we developed the activity shown in figure 5.

After the groups have worked on the activity, some students present their ideas of why the diagrams constitute a proof. When this occurs, all of the students are already engaged in the problem and ready to listen to the reasoning. In general, many students will have missed the fact that they need to prove that the shaded figure in the diagram on the right is a square and not merely a rhombus. But because students have been engaged, they will listen to the explanation of a peer concerning that fine point. If no student has clarified this, the teacher can raise the issue and lead the students, through a sequence of questions, to see why the shaded figure is a square. Teachers report that this whole-class discussion has been much more successful than discussions that occurred before the activity in figure 5 was included.

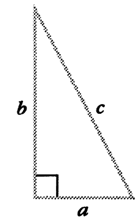
What about the students who had not tuned out originally? Why are they having to go through an activity when they could have learned the proof from a whole-class discussion? Teachers report that their "top" students also benefit from the activity, because they get the satisfaction and intellectual stimulation of developing the proof, rather than simply being told about it. High-achieving students are not penalized when a whole-group discussion is replaced by a small-group activity as long as the substituted activity is sufficiently interesting and challenging.

Why not individualized instruction?—The example above illustrates that discussions in small groups are generally easier for students to attend to than whole-class discussions. In the small-group format, each individual can participate easily and ask questions when she or he does not follow the group's train of thought. The pace of each small group is more likely to be suited to the individual than is the pace of an entire class.

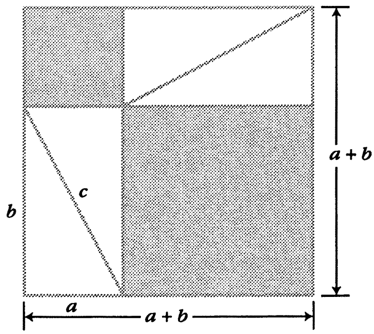
This raises the question of why not go even further and have students work as individuals, in which case each student can set his or her own

Proof by Rugs

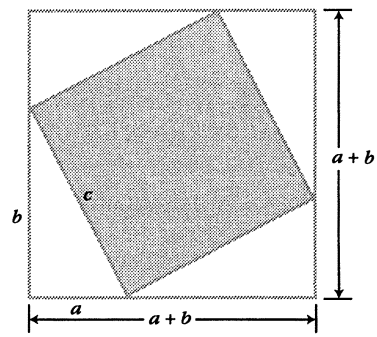
Al and Betty have another game. They began with this right triangle, which has legs of lengths a and b and a hypotenuse of length c. Then they made the two square rugs shown below. Each rug has sides of lengths $a + b$, and the triangles within each square are the same as the single right triangle shown at the right.



When it's Al's turn, a dart drops on the square rug on the left. If it hits the shaded area, he wins a point. When it's Betty's turn, the dart falls on the square rug on the right. If it hits the shaded area, she wins a point. Assume that the darts always hit the rugs, but that they land randomly within the rug. In other words, all points on a rug have the same chance of being hit.



Al's Rug



Betty's Rug

1. Is this a fair game? That is, is the chance of the dart landing on the shaded area the same for the two rugs? Explain your answer.
2. How do the two rugs demonstrate that the Pythagorean theorem holds true in general?

FIG. 5

pace. Indeed, individualized seat work has been a traditional answer to the need for involvement. But individualized seat work cannot involve students as well as small-group work does in the tasks of investigating, conjecturing, formulating, and proving. Those activities are best done in a group, where there are a variety of ideas to lend depth to the process.

Providing for individual differences.—If a broad range of students are to be served by a single curriculum, there must be provisions for individual differences. The Interactive Mathematics Program provides a large collection of supplemental activities for this purpose. These activities are of two types: reinforcements, which allow students to strengthen their understanding of material in a given unit, and extensions, which go beyond the basic material and are often more abstract or involve proof or generalization.

Both types of supplemental activities need to be engaging, although they concentrate on different ideas. Figure 6 shows a set of reinforcement activities for students who need more work on the idea of area measurement, while figure 7 shows an extension on area measurement, involving the Pythagorean theorem, that requires extensive organizational skills.

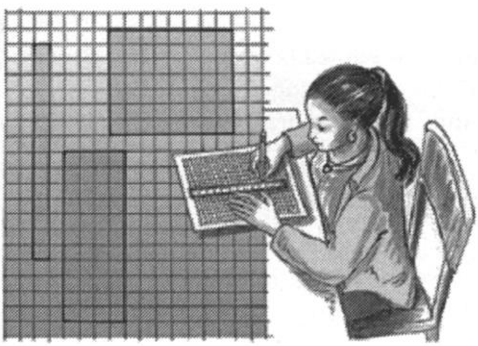
Teachers can sometimes use these activities as homework, giving different students different assignments or allowing students to choose different assignments. Teachers can also use the supplements during class, giving groups different assignments or letting each group choose the kind of activity they find most appropriate. Even though different individuals in a group have different levels of understanding, the group can work together effectively on a task, with everyone benefiting if it is engaging.

Students Need a Reason for Doing Problems

Since IMP's aim is to empower students mathematically, we are actually expecting students to work harder in mathematics classes than they have in the past. To attain mathematical power, they have to think, which is hard work. To get students to expend this extra effort, curriculum writers must provide sufficient motivation.

There are many students who will work on problems when given very general motivation such as "It will be on the test," "You'll need it in math next year," "You'll need it in college," or, even more vague, "It's important to know this." In contrast, many students need more intrinsic motivation for working on mathematics. They may not see themselves as going to college or as needing any mathematics in their future jobs. Educators need to reach these students as well as those who accept the idea that mathematics is a part of their future.

Measurement Medley —Area



Area is generally measured in "square units" of some kind. In this activity, you'll work with square centimeters, square kilometers, and square inches.

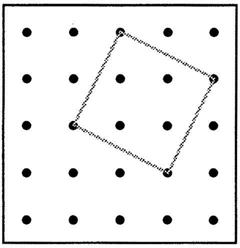
1. Use centimeter grid paper, a globe, and the fact that the distance around the earth at the equator is about 40,000 kilometers to estimate the area of the continental United States in square kilometers.
2. Suppose that 0.1 ounce of a certain skin cream will cover 50 square inches of skin. Estimate how much cream you would need to cover your arm.
3. Use centimeter grid paper for these tasks and questions.
   a. Draw at least five different rectangles, each with a perimeter of 30 centimeters.
   b. Which of your rectangles in Question 3a has the smallest area? The largest area?
   c. Draw at least five different rectangles, each with an area of 24 square centimeters.
   d. Which of your rectangles in Question 3c has the smallest perimeter? The largest perimeter?
   e. What conclusions can you draw from your work in Questions 3a through 3d?

FIG. 6

Geoboard Squares

In *Checkerboard Squares* (a Problem of the Week from the Year 1 unit *Patterns*), you were asked to find the number of squares (of any size) on an 8-by-8 checkerboard. In that problem, the squares all had vertical and horizontal sides, because they were made of one or more individual colored squares from the checkerboard.

On a geoboard, however, there are other kinds of squares. For example, the geoboard shown here shows a square with slanted sides.



Your task in this activity is to find out how many squares ***of all kinds*** there are on a geoboard. The only condition is that the vertices of the squares must be at pegs. Don't just find out how many ***kinds*** of squares there are. Consider every square on the geoboard as an individual example.

Caution: Not every four-sided figure is a square. Be sure that all the figures you include are squares.

Start with the standard 5-peg-by-5-peg geoboard, and then consider geoboards of other sizes. Look for a method for calculating the number of squares on an n-peg-by-n-peg geoboard.

Extra: Generalize your results to an m-peg-by-n-peg rectangular geoboard.

FIG. 7

We have found that students are willing to put out the required energy if they think the task at hand has genuine value, although their reasons for valuing the task may vary from task to task and from student to student.

Real-life contexts.—One major method for motivating students to work in mathematics class is to put problems in context. Students will value a task if they believe it relates to their real lives, present or future. For example, in the activity shown in figure 8, students study the consequences of mandatory drug testing.

Other contexts that students find meaningful include solving a linear programming problem to maximize profit for a business and developing a calculator program to generate a graphical display that moves on the screen. Although students recognize that they might not face these precise challenges in the future, the tasks have a feel of reality or relevance that maintains their interest.

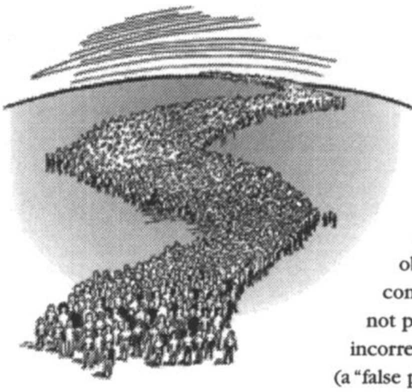
Capturing students' imagination.—Students also value a task when it catches their imagination. One problem we have used involves Edgar Allan Poe's story "The Pit and the Pendulum," in which a man escapes from the deadly swinging blade of a 30-foot pendulum. Students are asked whether the amount of time the prisoner needs for his escape in the story fits the reality of how pendulums work. After experimenting with smaller pendulums and extrapolating from their data through curve fitting, students actually build a 30-foot pendulum to check their mathematical analysis.

Another problem asks students to figure out when someone should jump from a moving Ferris wheel to land in a moving tank of water. They are highly motivated to do the complex mathematics required to create a "splash" instead of a "splat."

Tapping into intellectual curiosity.—A third reason why students value a task is that the mathematics itself is intriguing. One problem IMP uses asks students to investigate which numbers can be written as linear combinations of other numbers. But while the mathematics of this problem eventually engages the students' attention, we have found that students often need a context for such problems to get them started. That is, their intellectual curiosity often is not activated until they begin working on the problem. It is not a lack of motivation that makes students hang back from the task. Rather, as we learned from experience, many students find it hard to understand a task when it is stated very abstractly. A context—even a far-fetched one—can make the situation concrete enough for students to begin thinking about the problem. The context we have used for the linear-combinations problem involves a football scoring system. The activity is shown in figure 9.

Homework 12

Drug Dragnet: Fair or Foul?



People in many occupations, such as police officer or air traffic controller, are subject to random drug testing, on the grounds that their work affects the safety of the general public.

However, there is considerable controversy about such testing. One objection concerns the potential unfair consequences of the fact that the tests are not perfect. For instance, if the test incorrectly shows someone to be a drug user (a "false positive"), that person could lose his or her job. This assignment explores some of the mathematical issues in drug testing.

In this assignment, you are to assume that a certain test for drug use is 98% accurate. By this, we mean that 98% of the people who use the given drug within some specified time period will test positive and 98% of the people who did not use the drug in that time period will test negative. Also assume that only 5% of the people on the job (1 in every 20) engage in drug use.

1. If someone tests positive, how likely is it that the person has actually engaged in drug use within the given time period? (*Hint:* Consider a large population, such as 10,000 people. Figure out how many use drugs and how many users and nonusers test positive.)
2. Do you think a test such as this should be used? Explain.

FIG. 8

POW 12 Kick It!

The Free Thinkers Football League simply has to do things differently. The folks in this league aren't about to score their football games the way everyone else does. So they have thought up this scoring system:

- Each field goal counts for 5 points.
- Each touchdown counts for 3 points.

The only way to score points in their league is with field goals or touchdowns or some combination of them.

Continued on next page

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FIG. 9

One of the Free Thinkers has noticed that not every score is possible in their league. For example, a score of 1 point isn't possible, and neither is 2 or 4. But she thinks that beyond a certain number, all scores are possible. In fact, she thinks she knows the highest score that is impossible to make.

1. Figure out what that highest impossible score is for the Free Thinkers Football League. Then explain why you are sure that all higher scores are possible.
2. Make up some other scoring systems (using whole numbers) and see whether there are scores that are impossible to make. Is there always a highest impossible score? If you think so, explain why. If you think there aren't always highest impossible ones, find a rule for when there are and when there are not.
3. In the situations for which there is a highest impossible score, see if you can find any patterns or rules to use to figure out what the highest impossible score is. You may find patterns that apply in some special cases.

Write-up

1. *Problem Statement*
2. *Process:* Include a description of any scoring systems you examined other than the one given in the problem.
3. *Conclusions:*
   a. State what you decided is the highest impossible score for the Free Thinkers' scoring system. Prove both that this score is impossible and that all higher scores are possible.
   b. Describe any results you got for other systems. Include any general ideas or patterns you found that apply to all scoring systems, and prove that they apply in general.
4. *Evaluation*
5. *Self-assessment*

FIG. 9 (*continued*)

Although it is important for students to learn to understand statements in pure mathematics, curriculum writers must recognize that many of them need to start with more concrete situations and build from there. Such work will undoubtedly help them to understand abstractions in the future. Once students do get involved in problems like these, it is the mathematics—not the context—that holds their attention.

Final Conclusions

Our primary goal in IMP was to design curriculum rather than to prove any general principles. We conclude the article, however, with some information about the program's outcomes, some political consequences of our activity, and some challenges for further research.

Evaluation of the Approach

What happens to students in a curriculum that tries to meet the needs of all students? The Interactive Mathematics Program began in 1989 and is now well beyond its pilot stage both in terms of curriculum development and its implementation in schools nationwide, and we can offer quite a bit of data on its success.

IMP students hold their own on traditional measures.—Using traditional standardized tests, such as the Scholastic Achievement Test (SAT), many different schools in different parts of the country have compared IMP students with students in the traditional sequence. In all of these studies, there has either been no significant difference in scores or the IMP students have scored significantly higher. Figure 10 shows data from Eaglecrest School in Aurora, Colorado, that compares ninth-grade IMP students with ninth-grade Algebra I students at the beginning of their IMP and algebra experience and at the end of the year.

The consistency of these studies shows that IMP students are holding their own on these traditional measures. Thus, they are not being harmed by a nontraditional curriculum. This finding is especially meaningful in view of the fact that about 20–30 percent of the IMP students' classroom time is spent learning important topics not covered on these standardized tests.

Achievement beyond standardized tests.—Norman Webb from the Wisconsin Center for Educational Research began a five-year evaluation of the project in 1992. Part of his study was a transcript analysis of the 1,121 students who graduated in 1993 from our three original pilot schools. Here are two major findings from that study:[14] (1) a higher percentage

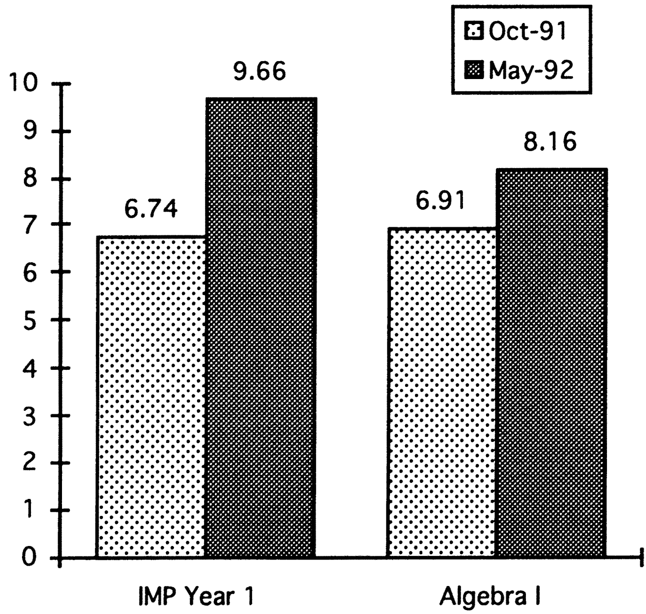

FIG. 10.—SAT average raw score, Eaglecrest High School, Aurora, Colorado. Average raw scores of IMP Year 1 students increased by 2.92, while average raw scores of Algebra I students increased by 1.25. The difference in growth is significant at the .025 level.

of students (90 vs. 74 percent) in the IMP program took at least three years of college-preparatory mathematics than did students enrolled in the traditional Algebra-Geometry-Algebra II sequence, and (2) IMP students consistently achieved higher grade point averages than did students in the traditional program, both in mathematics and overall. These findings illustrate important ways in which students are being helped.

In spring 1996, Webb conducted three separate studies of mathematical achievement using nontraditional measures. These studies involved different grade levels and different schools. In all three studies, IMP students significantly outperformed students using a traditional curriculum.

The ninth-grade study, involving 60 Year 1 IMP students and 50 Algebra I students, used the statistics items from the Second International Mathematics and Science Study. Eighth-Grade Comprehensive Test of Basic Skills scores in math were used as a covariate. The mean of IMP students was 3.1 out of a possible 5, while the mean of the Algebra I students was 1.0.

The tenth-grade study, involving 53 Year 2 IMP students and 66 Geometry students, used two constructed-response tasks from a performance assessment developed by the Wisconsin Student Assessment System. The tasks required students to solve two multistep mathematics problems, generalize the results obtained, and write an explanation of their procedure. Although there was no difference in the means of the groups on their eighth-grade Iowa Test of Basic Skills, the IMP students had a mean of 6.44 on the performance assessment (out of a possible 10), compared with a mean of 4.56 for the geometry students.

The eleventh-grade study, involving 93 Year 3 IMP students and 40 Algebra II students, used items from a quantitative-reasoning test given to all incoming freshmen at a prestigious eastern university. The study used eighth-grade standardized mathematics test scores as a covariate, and the IMP students outperformed the Algebra II students with a mean of 5.04 (out of 10) versus 2.40.

Impact at both ends of the spectrum.—The findings that IMP students take more mathematics and achieve higher overall grade-point averages apply to the full spectrum of the student population. There is also evidence that IMP's approach is helpful to students who score at both ends of the achievement spectrum. For example, in a study of high schools with low-income and low-achieving student populations, higher growth in achievement was observed among ninth-grade students enrolled in the IMP courses than was observed for those enrolled in the other curricula, including traditional algebra (White et al. 1995). The IMP students began at a lower level than the other students in college-preparatory courses and finished at a higher level.

There is also data on the positive effect of IMP on students who had been high achievers in traditional programs. At Central High School in Philadelphia (an examination school for high-achieving students), eleventh-grade IMP students had significantly higher scores than their counterparts from a traditional curriculum on both mathematics and verbal PSAT tests.[15]

At one of the original IMP pilot schools, transcripts included students' seventh-grade scores on the Comprehensive Test of Basic Skills. Webb's work included a study of those students who had scored in the upper 25 percent on this test on the basis of national norms. Within this category of "high-achieving" students, he compared the IMP group ($N = 58$) with the non-IMP group ($N = 60$). On completion of high school, the IMP students had a higher percentage taking the SAT (83 vs. 72 percent), higher mean scores for those who took the SAT (545 vs. 531), a higher percentage scoring 600 or above (31 vs. 21 percent), and a higher maximum SAT score (750 vs. 710). The IMP students also had a significantly higher overall grade-point average (3.11 vs. 2.68), even when mathematics grades were ignored (Interactive Mathematics Program 1996).

Political Consequences

One obstacle to the implementation of a single curriculum for all students written in the spirit of the NCTM *Standards* is the passionate opposition that arises from some parents and from some political groups. There appear to be at least two different kinds of opposition. One source of opposition is to a single curriculum taught to all students without tracking. The second kind of opposition is against the teaching techniques used in IMP and against the overall direction of mathematics teaching laid out by the NCTM *Standards.* The second type of opposition seems to involve complex issues that are outside the scope of this article, but we will briefly discuss the first type of opposition.

The opposition to a single curriculum for all students, without tracking, is expressed in a document produced by an organization calling itself Mathematically Correct:[16] "Differences in learning rates must be recognized and provided for. In most secondary schools the student population presents wide variations in student interest, aptitude and training with respect to mathematics and there is no point in pretending otherwise. Schools are to be commended for making curricular provisions for these variations. It is not unusual for our larger high schools to provide as many as six levels of instruction for grade nine. These are not established by faculties bent on making invidious comparisons. It is simply impossible to meet the needs of the entire school population without them" (Allen 1996, p. 1).[17]

This quotation illustrates the conventional wisdom that high levels of learning cannot occur throughout very heterogeneous classes. On the one hand, there is a strong feeling that students who are not succeeding in traditional classrooms need the material handed to them more slowly in order to prevent failure. On the other hand, it is felt that a single untracked curriculum will hold back the most successful students. These views persist despite evidence from programs, such as IMP, that successful learning can occur in heterogeneous classes. In fact, there is little evidence that tracking helps high-achieving students and considerable evidence that tracking does great harm to low-achieving students.[18]

The feeling that both low and high achievers are being deprived by the reform movement is one rationale given for separate curriculums. For example, one newspaper columnist, who often writes against the math reform movement, said "Write this in stone: Poor and bilingual kids need more basics—not less" (Saunders 1995). In fact, she and many other people feel that any curriculum that serves all students must be "dumbing down" the material (Saunders 1997).

A three-year study (Wells and Serna 1996) looked at the implementation of detracking in 10 schools across the country and focused on opposition to the efforts from parents. The original intent of the study was to examine what happened inside the school, but they soon found out that "the parents had a major impact on detracking efforts" (Wells and Serna 1996, p. 116). In particular, they found: "The assumption here was that if there was no selectivity in placing students in particular classes, then the learning and instruction in those classes could not be good" (Wells and Serna 1996, p. 102).

When schools first began to use our new curriculum, we asked them to maintain a traditional program so that parents and students would have a choice. But some schools chose to ignore our advice and offered only the IMP curriculum. Almost all of the schools that did not offer choice met with strong opposition from parents.[19]

Areas for Research

We conclude this article with some questions that others may wish to pursue:

1. Is a single core curriculum for all students the best design to meet both the needs of individual students and the overall needs of society?
2. What is the effect on student learning of the decision to use problem contexts that may appeal more strongly to some students than to others?
3. Is the emphasis on collaborative learning sufficiently "culture neutral" as to be appropriate for all students?

Notes

1. This goal is articulated, e.g., on p. 2 of *Mathematics Framework for California Public Schools* (California Department of Education 1992).

2. This program received major funding from the National Science Foundation under award number ESI-9255262. Any opinions, findings, and conclusions or recommendations expressed in this article are those of the authors and do not necessarily reflect the views of the National Science Foundation.

3. Though we say "all" here, we acknowledge that there is probably a small percentage of students at each end of the spectrum that have special needs that cannot be addressed by any regular program. The word "all" should perhaps be interpreted as the middle 95 percent.

4. One of the primary failings that we have observed in the tracked approach is that students who are placed in a lower track because of poor performance are

often students who simply have a nontraditional style of learning and doing mathematics. Students who do poorly at the early grades are often capable of high-level work, but, once tracked, they are often not provided an opportunity to enter the higher-level track.

5. Some of the programs that the authors have worked with are The Madison Project, Project SEED, and EQUALS. For information on The Madison Project, see Davis (1980). For Project SEED, see Clewell et al. (1992). For EQUALS, see Fraser (1982).

6. By no means do we claim exclusive ownership or originality of these principles. In general, the curriculum development projects funded by the National Science Foundation during the 1990s have shared many of the same philosophical and pedagogical underpinnings.

7. In a letter to the syndicated Miss Manners column (Martin 1996), a mathematics instructor commented that when she told someone that she liked teaching mathematics, the listener contorted her face "as if I had just told her I enjoyed eating worms."

8. See, e.g., Hankes (1996).

9. For example, the Mathematical Association of America's excellent program, Women and Mathematics, identifies four functions, none of which includes making mathematics courses more hospitable to young women.

10. See, e.g., Seegers and Boekaerts (1996).

11. Rationales for group work are summarized in Johnson and Johnson (1990).

12. For example, Tom Ferrio, business manager, Calculators for Education, for Texas Instruments, has written to us in a personal communication (1996), "The ability to solve technical problems while working in a team is an essential attribute of employees in the modern workplace. The ability to work with others is often a more important hiring criteria than raw talent."

13. For information on the background and implementation of this approach, see Davis et al. (1990). Note that active learning often goes hand in hand with students constructing their own meaning, although these approaches to learning are not the same.

14. For more details, see Interactive Mathematics Program (1996). This publication is available from the IMP office (1-888-MATH-IMP).

15. The study included all eleventh-grade students who, as freshmen, enrolled in either IMP 1 ($N = 99$) or Algebra I ($N = 325$). For mathematics, the mean scores (reported in three-digit SAT fashion) were 544 for the IMP group and 522 for the algebra group. For the verbal test, the mean scores were 551 for the IMP group and 529 for the algebra group. For more information, see Interactive Mathematics Program (1996).

16. This organization involves parents and educators who are concerned about what is being called "the new-new math" and who want to reemphasize basic skills. The organization is politically active in California. For more information, write to Mathematically Correct, P.O. Box 22083, San Diego, CA 92192.

17. The author, Frank Allen, is an advising member to Mathematically Correct.

18. This result is true when tracked students are compared to comparable untracked students who are studying the same material (Slavin 1990).

19. Some opposition to IMP has even come from parents whose children were not in IMP classes. There is some indication that their opposition stems from a general objection to the use of heterogeneous classes.

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