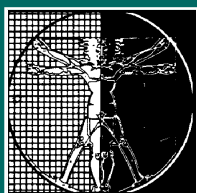


3



Prediction

LESSON ONE

The Hip Bone's Connected...

LESSON TWO

Variability

LESSON THREE

Linear Regression

LESSON FOUR

Selecting and Refining Models

Chapter 3 Review





PREDICTION

W

ould you like to predict the future or solve a past mystery? Equipped with a crystal ball or time machine, what information would you predict? The number of U.S. gold medals at the next Summer or Winter Olympics? Your salary after working 10 years? Your grade point average when you graduate? Your great, great grandfather's height?

Alas, you don't have a crystal ball or time machine. That doesn't mean, however, that you can't try to predict future outcomes or solutions to past mysteries. Economists predict unemployment; the government predicts its tax revenues; and businesses predict sales. Anthropologists "predict" features of early humans. Forensic scientists make predictions about assailants and murderers.

Instead of crystal balls or time machines, all of the people mentioned above use real data. They build mathematical models from their data. Their models help them make useful predictions. In this chapter, you examine patterns in data. You learn to draw a line that best represents the data. Then you use the equation of your line to make predictions. In the process, you extend your understanding of lines and the equations that describe them.



LESSON ONE

The Hip Bone's Connected...

Key Concepts

Explanatory and response variables

Residual error

Linear models

Graphs of lines

Solving linear equations

Scatter plots

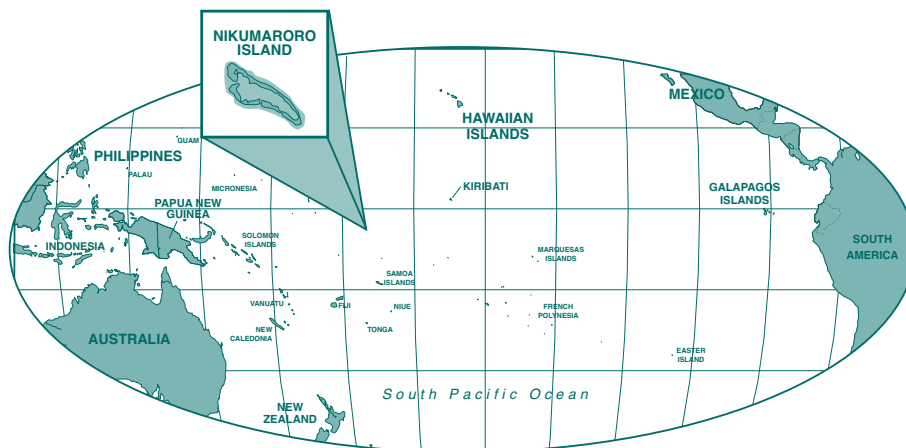
PREPARATION READING:

The Disappearance of Amelia Earhart

Amelia Earhart was the first woman to fly solo across the Atlantic Ocean. Later, she flew across the Pacific. In 1937, she and her navigator Frederick Noonan set off on a flight around the world. Their plane disappeared, and their fate remains a mystery to this day.



The International Group for Historic Aircraft Recovery (TIGHAR, pronounced tiger) has investigated the disappearance for more than 15 years. TIGHAR believes that Earhart crashed on the tiny island of Nikumaroro in the Republic of Kiribati. In 1997, TIGHAR discovered papers in Kiribati's national archives. These papers reported a 1940 discovery of bones. The bones were sent to a medical school in Fiji. There they were examined by Dr. D. W. Hoodless. He concluded that the bones were from a male about 5 feet 5 inches tall.



Statements in Dr. Hoodless' report raised doubt about his knowledge of the human skeleton. Although the bones have disappeared, Dr. Hoodless' report contained the measurements that he took (see Figure 3.1).

Bones	Length (cm)
Humerous	32.4
Tibia	37.2
Radius	24.5

Figure 3.1.
Some bone measurements
taken by Dr. Hoodless.

Recently, forensic anthropologists Dr. Karen Burns and Dr. Richard Jantz analyzed these measurements. They concluded that the bones were from a white female of European background who was 5 feet 7 inches tall. This description fits Earhart.

Forensic anthropologists use data to build mathematical relationships between two (or more) quantities. Then, they use these relationships to make predictions. From measurements of skeletal remains, they predict such things as height, gender and ethnicity. In this lesson, you will learn how to use relationships between two quantities to make predictions.

DISCUSSION/REFLECTION

1. How could data on bone lengths and heights give scientists clues about the person whose bones are described in Figure 3.1?
2. Name some fields of work where data are collected and then used to make predictions.

Activity 3.1: Using Your Head

Scientists often use mathematical models to help investigate human remains. In this activity you will sort through a group of bones looking for clues. You will explore models of the form $y = mx$ that describe the relationship between the length of a person's head and the person's height. Then you will use one of these models to make a prediction.

MYSTERIOUS FINDINGS

From time to time, bones are found in rugged areas. A hiker in Arizona's rugged Superstition Mountains finds a skull, eight long bones, and numerous bone fragments. He notifies local police who send a team of specialists to investigate. The team records information about the bones, such as their size and general condition. **Figure 3.2** shows information similar to what the team might have recorded.

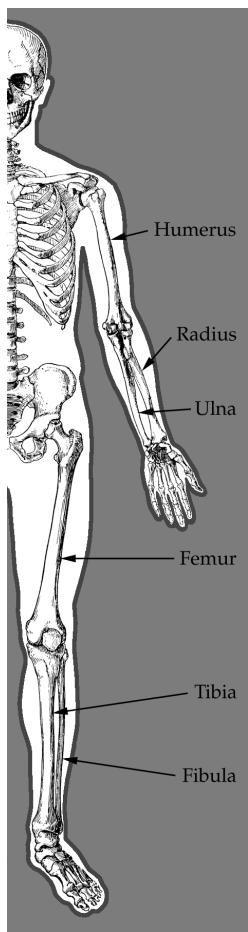


Figure 3.3.
A human skeleton.

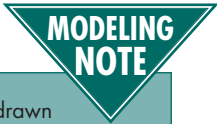
Bone type	Number found	Length (cm)
Femur	3	41.4, 41.5, 50.8
Tibia	1	41.6
Ulna	2	22.9, 29.0
Radius	1	21.6
Humerus	1	35.6
Skull (with jawbone)	1	23.0
Fragments	More than 10	3.0–5.0

Figure 3.2.
Sample record of bones found at site.

FYI

At birth you had more than 300 bones in your body. As you grow, some bones fuse together and you wind up with about 206. The smallest bone is the stirrup bone. It's in your ear. The largest bone is the femur. It's in your leg. You have different kinds of bones in your body. Long bones are shaped like a tube. They can be found in your fingers, arms, and legs. Short bones are wide and chunky. They're in your feet and wrists. Flat bones are flat and smooth. Your ribs and shoulder blades are flat bones. Irregular bones have odd shapes. The bones in your inner ear and vertebrae are irregular bones.

1. a) Study the data in Figure 3.2. The team reports that these bones belonged to at least two people. How do they know for sure?
 - b) Which bones do you think belonged to the same person? Explain how you arrived at your answer. (To make it easier to classify the bones, refer to the dead people as Skeleton 1, Skeleton 2, and so on.)
 - c) Do you think the deceased were young children or adults? Defend your answer.
2. a) Guess the heights of the dead people whose bone measurements are in Figure 3.2. Explain how you got your answer.
 - b) How precise do you think your guesses are: Within a foot of the true heights, within 6 inches, or within 1 inch?



Have you ever drawn a sketch of a person and found the head looked the wrong size for the body? Artists face the same problem. Based on drawings and knowledge of human anatomy, they proposed a solution, the formula “height = 7 head lengths.” Formulas that are solutions to modeling problems are often called models. But keep in mind that the modeling process is more than the formula. It begins with a problem or question. Often involves gathering data or information. Then uses mathematics to turn that information into a solution.

A MODEL FOR ESTIMATING HEIGHT

One place to look for help in estimating heights is an artists’ guidebook. Artists have found that the rule of thumb, “a 14-year-old is about 7 head-lengths tall” helps them draw teenagers with heads correctly proportioned to their bodies.

3. First you collect some data. Your data will help you decide how closely real students match the model suggested by artists.
 - a) Within your group, measure each person’s head length (from chin to the top of the head). Record your data in the first two columns of a table like **Figure 3.4**. (Leave room to add an additional column.) Be sure to specify your units of measurement for head length at the top of the second column.

Name	Head length	Actual height	Predicted height

Figure 3.4. Group head-length and height data.

- b) Measure each person’s actual height. Then record your results in the third column. Be sure to record your units for height at the top of the column.

- c) Use the relationship of height = 7 head lengths to predict each person's height in your group. Record your results in the fourth column of your table. Be sure to record the units for predicted height at the top of the column.

The artists' guideline "height is seven times head length" uses head-length measurements to explain height measurements. Therefore, head length is called the **explanatory variable**. In turn the height measurement responds to changes in the head-length measurement. Therefore it makes sense to call height the **response variable**.

How good was the artist's model, height = 7 head lengths? One way to evaluate the model is to calculate what are called **residual errors**.

When a prediction is made from a model, the difference between the actual value and the predicted value is called the **residual error**.

Residual error = actual value – predicted value

4. a) Add a fifth column to your table and label it "Residual error." Calculate the residual errors for your data in Question 3. Subtract the predicted height of each student in column 4 from the actual height in column 3. (Make sure your units agree!) Record your results in column 5.
 - b) You will need enough data to recognize patterns. Collect data from at least one other group. Add these data to your table.
 - c) If a residual error is positive, does the prediction overestimate or underestimate the actual height?
 - d) What if a residual is negative? What if a residual is zero?
5. a) Look at the residual errors in your table from Question 3. Are there an equal number of positive errors and negative errors? How many of the errors are positive? How many are negative?
 - b) How well did the model "height = 7 head lengths" do in estimating heights of students in your class?
 - c) Your next task is to find a better model. Adjust the model in (a) by changing 7 to a different number. What number did you choose? Explain how you found it.

TAKE NOTE

Sometimes the term residual error is shortened to residual or error.

You have a model: height = $(7)(\text{head length})$. The next step in the modeling process is to decide how well the model works. One way to judge a model is to look at the residual errors. If the residuals are mostly positive, then the predictions from your model tend to be too small. If the residuals are mostly negative, then the predictions tend to be too large. In either case, you'll need to adjust your model. Try to find a new model (formula) which has residual errors that are nearly equally divided between positive and negative values.

- d) Use your model from (c) to complete a table like the one in Figure 3.4. Add a fifth column for the residual errors.
- e) What evidence do you have that your model is better than the model in (b)?

MODELS OF THE FORM $Y = MX$

To graph the artists' model of height = 7 head lengths using your calculator, you enter the formula $y = 7x$. Artists have found that this formula works fairly well for drawing 14-year olds. However, the relationship between height and head length changes with age. Therefore, artists adjust their guideline based on the age of the person they are drawing. When drawing sketches of adults (ages 18–50), artists follow this guideline:

Adults are about $7\frac{1}{2}$ head-lengths tall.

This model can also be written in the form $y = mx$. All that changes is the value of m . Instead of using 7 for m , use 7.5.

6. a) Using the artists' guideline for adults, predict the height of a person whose head length measures 23.0 cm.
- b) Without doing any calculations, would your estimate be higher or lower if you knew the person was only 13 years old? Explain.

So far, you have explored a few models that describe the relationship between height and head length:

- models based on artists' guidelines for drawing figures
- a model based on data collected by one or more groups

7. Think how you might use one of these models to make a rough prediction of the height of the person whose skull length was recorded in Figure 3.2.
- a) What assumptions might you make in order to make your prediction?
- b) Recall that the skull measured 23.0 cm in length. Predict the height of the person in centimeters (cm). Describe the process you used in making your prediction.

c) Does your prediction result in a height that is reasonable for a person? (Recall that 1 inch = 2.54 centimeters.) Explain.

d) Do you think your prediction is likely to be close to the actual height of the person? Why or why not?

Activity Summary

In this activity, you:

- ♦ examined and interpreted models (equations or formulas) that predict height.
- ♦ gathered data and used it to find a formula to predict height based on the length of a person's head.
- ♦ used the residual errors to compare actual and predicted values. The residual errors helped you decide if you had a good model.

These models give very rough estimates of a person's height. In later activities you will develop models that give more precise predictions.

DISCUSSION/REFLECTION

An artists' model for drawing 14-year-olds is: $\text{height} = (7)(\text{head length})$.

1. A toddler's body, however, is proportioned differently than a 14-year-old's. Would you change the 7 to a larger or smaller number to produce a guideline for drawing toddlers? Explain.
2. Suggest a model that you think might work well for drawing toddlers.

An artists' model for drawing adults (18–50) is: $\text{height} = (7.5)(\text{head length})$.

3. Unfortunately, bodies change proportion again as people age. Propose a model for drawing 80-year-olds. Explain how you decided on your model.
4. How would you check how well your proposed models in 2 and 3 work?

In this assignment you will work with models of the form $y = mx$ and $y = mx + b$. Adding the constant b to the models will allow you to make more precise predictions. In addition you will adapt the form of these models by solving for one variable in terms of another.

1. In Activity 3.1, you worked with artists' guidelines for drawing 14-year-olds and then for drawing adults:

Draw 14-year-olds seven head-lengths tall.

Draw adults seven-and-a-half head-lengths tall.

You can treat these guidelines the same way you did coding processes. Here, the coding process stretches head-length measurements and turns them into height measurements.

- Draw an arrow diagram that represents the artists' guideline for drawing 14-year-olds.
- Draw an arrow diagram that represents the artists' guideline for drawing adults.
- Suppose you know that an adult is 63 inches tall. Predict the person's head length. (Hint: Decode the height.)
- Suppose you know a 14-year-old is 63 inches tall. Predict the teenager's head length.

For Question 1(a) and (b) you drew two arrow diagrams. Each of your arrow diagrams shows that you calculate a figure's height by multiplying its head length by a number, m . Your diagrams represent formulas that have the form $y = mx$, where y is height and x is head length. That is, y and x are **proportional**, and m is the **constant of proportionality**.

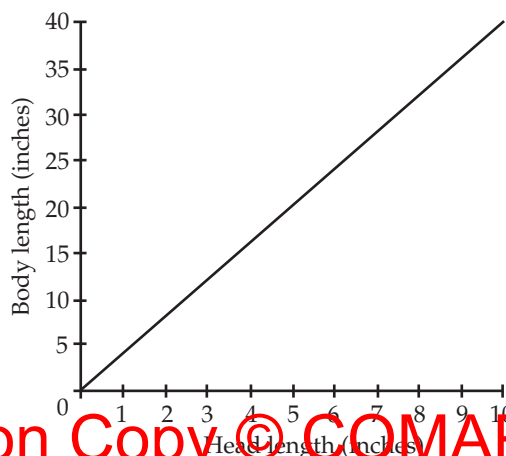


Figure 3.5.
Graph of artists' guideline for drawing babies.



You have seen the words

proportional and constant of proportionality before in Activity 2.5, Chapter 2, *Scene from Above*. There you learned that two variable quantities are proportional if their ratios never change. If $y = mx$, then the ratio $y/x = m$ and hence this ratio is constant.

2. The graph in Figure 3.5 represents the artists' guideline for drawing newborn babies.
- Using the graph, predict the height (which is really body length since babies can't stand) of a newborn whose head length is 5 inches.
 - Write a formula for the artists' guideline for drawing babies.
 - The proud parents of a newborn announced that their baby weighed 8 pounds and was 22 inches long. Predict the length of the baby's head.

TAKE NOTE

The bone measurements in Figure 3.2 were measured to the nearest tenth of a centimeter. Any predictions that you make from these bones can only be as precise as the measurements on which they are based. So after you calculate your height predictions for 3(b), round them to one decimal place.

3. In the late 1800s, models for predicting height from the length of long bones were based on ratios. (For a refresher on the long bones, see Figure 3.3.) One model is given below:

$$\frac{H}{F} = 3.72, \text{ where } H \text{ is height and } F \text{ is femur length.}$$

- Which is the explanatory variable? Which is the response variable? Explain how you decided.
- Based on this model, estimate the heights of the people whose femur lengths are given in Figure 3.2, Activity 3.1.
- Suppose, for most adults, femurs range in size from 38 cm to 55 cm. According to this model, what is the range of heights of most adults? Are these reasonable heights for adults? (Recall that 2.54 cm = 1 in.)
- What formula would you enter into your graphing calculator in order to graph this relationship between height and femur length?

The formula that you wrote for 3(d) should be a member of the $y = mx$ family. Dr. Mildred Trotter (1899–1991), a physical anthropologist, was well known for her work in the area of height prediction based on the length of the long bones in the arms and legs. She refined earlier models by adding a constant, thereby producing models of the form $y = mx + b$.

Here is one of Dr. Trotter's models. $H = 2.38F + 61.41$

where H is the person's height (cm)

and F is the length of the femur (cm).

Questions 4–9 are related to Dr. Trotter's model relating height and femur length.

- Assume that most adults' femurs range in size from 38 cm to 55 cm.
 - According to Dr. Trotter's formula, how tall is a person with a 38-cm femur?
 - How tall is a person with a 55-cm femur?

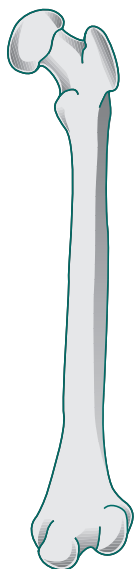


Figure 3.6.
The femur (thighbone).

In Dr. Trotter's formula, femur length is used to predict height. Therefore femur length is the explanatory variable. Height is the response variable. To draw a graph of this relationship, you should use the horizontal axis to represent the explanatory variable. The response variable goes on the vertical axis. In the next question you will draw a graph of Dr. Trotter's relationship.



When graphing a model, use the horizontal axis for the explanatory variable. Use the vertical axis for the response variable. Remember to label the axes with the names of the variables and, when appropriate, the units of measurement.

5. a) On graph paper, draw a set of axes similar to that shown in **Figure 3.7**.

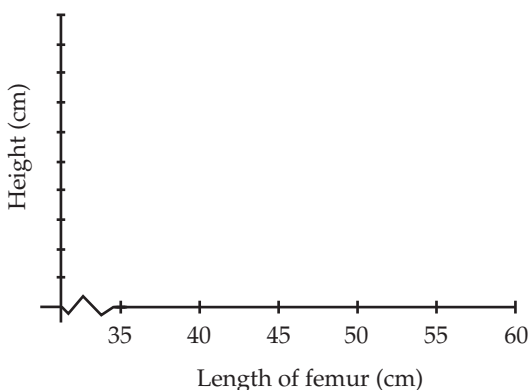


Figure 3.7.
Axes for height and femur length.

Notice that the horizontal axis is scaled from 35 cm to 60 cm (a slightly wider range than the minimum and maximum femur lengths) with tick marks every 5 units. A zigzag has been added to indicate there is a break in this scale between 0 and 35.

- b) Draw a scale on the vertical axis that would be appropriate for data on adult heights. Add a zigzag to the vertical axis if you break the scale between 0 and another number.
- c) Sketch a graph of Dr. Trotter's formula, $H = 2.38F + 61.41$, on your axes. (You may want to plot several points before drawing the graph.)
6. Jason's femur measures 39 cm. His brother's measures 40 cm. Based on Dr. Trotter's formula, predict the difference in the two brothers' heights. (So that you can see a pattern, use two decimal places in your final answer.)
7. a) Suppose the femurs of two women differ by one centimeter. (In delta notation, $\Delta F = 1$.) Predict ΔH , the difference in their heights. (So that you can see a pattern, use two decimal places in your final answer.)
- b) Explain how you were able to determine your answer even though the lengths of the two women's femurs were not given.
- c) How could you read off your answer from Dr. Trotter's formula?

- d) Suppose instead, the femurs of two women differ by two centimeters, $\Delta F = 2$. Predict ΔH . Explain how you got your answer.
8. a) In Chapter 1, *Secret Codes and the Power of Algebra*, you drew arrow diagrams to represent a coding process. Draw an arrow diagram that represents Dr. Trotter's formula. Your arrow diagram should show how to predict height by "coding" femur length.
 - b) Now reverse the process in (a) and draw an arrow diagram that "decodes" heights. In other words, draw an arrow diagram that tells you how to use a person's height to predict femur length.
 - c) What formula does your arrow diagram represent?
 - d) Suppose a man is 178 cm tall. Use your formula in (c) to predict the length of his femur.

If you have done 8(c) correctly, your formula is algebraically equivalent to Dr. Trotter's model. That means that any height and femur-length pair that satisfy Dr. Trotter's model will also satisfy the equation that you wrote for 8(c).

Two equations are **algebraically equivalent** if they have the same solution.

9. a) Check that the pair $H = 168.51$ and $F = 45$ satisfies Dr. Trotter's formula. In other words, use your calculator to check that $168.51 = 2.38(45) + 61.41$.
- b) Check that the pair $H = 168.51$ and $F = 45$ satisfies your equation from 8(c). Show your calculations.

Another of Dr. Trotter's models predicts height from the tibia length:

$$H = 2.52T + 78.62,$$

where H is the person's height (cm)

and T is the length of the tibia (cm).

10. a) The length of the tibia described in Figure 3.2 was 41.6 cm. Use Dr. Trotter's formula, $H = 2.52T + 78.62$, to predict this person's height.
- b) Is your prediction in (a) a reasonable height for a person?
- c) Use your predicted height from (a) and your formula from 8(c) to predict the length of the femur of a person with the 41.6-cm tibia. Which femur length from Figure 3.2 is closest to your prediction?

In yet another model, Dr. Trotter used both the tibia and the femur to predict height.

$$H = 1.30(F + T) + 63.29 \text{ (All measurements are in cm.)}$$

11. Suppose that students measure the femur and tibia of a skeleton. They determine that the femur is 43.2 cm long and the tibia is 36.4 cm long.
 - a) Predict the height of the person using Dr. Trotter's formula $H = 1.30(F + T) + 63.29$. (What precision should you use in your answer?)
 - b) Predict the height of the person using Dr. Trotter's formula $H = 2.38F + 61.41$.
 - c) Predict the height of the person using Dr. Trotter's formula $H = 2.52T + 78.62$.
 - d) Compare your predictions from (a)–(c). Which do you think is closest to the person's actual height? Explain.
12. Use one or more of Dr. Trotter's models to estimate the heights of two of the people whose bones are described in Figure 3.2.
13.
 - a) Solve $H = 2.52T + 78.62$ for T . In other words, find an algebraically equivalent equation of the form $T = (\text{some expression})$. (Hint: Arrow diagrams may help.)
 - b) Solve $y = 5x - 6$ for x .
14. Are the two equations $H + 14 = 3L - 2$ and $H = 3L + 16$ algebraically equivalent? How did you decide?

Activity 3.2: Under Investigation

Dr. Trotter's equation $H = 2.38F + 61.41$ (where height, H , and femur length, F , are in cm) is a member of the $y = mx + b$ family. You indicate members of this family by choosing values for m and b . In this activity you will discover how your choices for m and b affect the graphs of $y = mx + b$ family members.

MODELING NOTE

Dr. Trotter's equation $H = 2.38F + 61.41$ was designed to work well for adult white males. She later modified her formula (by changing the constants 2.38 and 61.41) to adjust for racial/ethnic backgrounds and gender.

1. What were Dr. Trotter's choices for m and b ?

As part of the process of making predictions, you will often start with a model. Then later, in order to improve your predictions, you may need to adjust your model. For example, suppose you begin with a model from the $y = mx + b$ family. You may be able to improve your model by adjusting the values of m and b . But first, you'll need to understand how the control numbers m and b affect the graph.

2. Notice that there are two quantities to change, m and b . So, first simplify the problem. Set $b = 0$. Then, investigate how the control number m affects graphs of members of the $y = mx$ family.

- a) Use your graphing calculator to graph $y = 2x$. (The standard viewing window with $X_{\min} = -10$, $X_{\max} = 10$, $Y_{\min} = -10$ and $Y_{\max} = 10$ should work well.) Describe in words the graph of $y = 2x$.
 - b) In (a) the control number m has value 2. Vary m using values greater than 2. Describe how changing the control number in this way affects the graph. Illustrate by drawing graphs of several examples.
 - c) Now, chose positive numbers for m that are less than 2. Describe how changing the control number in this way affects the graph. Illustrate using several examples.
 - d) When you changed the control number m in (a)–(c), the graphs changed. However certain features of the graphs stayed the same. Describe what's similar about all of your graphs.
3. In your investigation of members of the $y = mx$ family in Question 2, you used only positive values for m . Select several negative values for m . How does changing the sign of the control number affect the graph? Illustrate by drawing graphs of several examples.

MODELING NOTE

A key strategy in modeling is to first simplify the problem. Solve the simplified problem. Then use your solution to the simplified problem to help you solve the original problem.

From your work in Questions 2 and 3, you should have a good idea how the control number m affects graphs of members of the $y = mx$ family. You've solved the simpler problem. Now it's time to return to the original problem: How does changing the control numbers m and b affect graphs of members of the $y = mx + b$ family?

TAKE NOTE

All members of the $y = mx$ family belong to the $y = mx + b$ family. That's because you can write them as $y = mx + 0$.

4. a) Start with $y = 2x + b$. How does changing the control number b affect the graph? Experiment using several values for b , including $b = 0$. Illustrate by drawing graphs of several examples.
- b) What features do all of your graphs in (a) have in common?
- c) Choose a different value for m . What is your value? Now graph $y = mx + b$ using your value for m and several choices for b . Choose both positive and negative values for b . What does b control? Illustrate using several examples.

In your work in Questions 2–4 you should have noticed that graphs of members of the $y = mx + b$ family are straight lines.

The graph of an equation of the form $y = mx + b$ is a straight line. The control numbers m and b are called **slope** and **y-intercept**, respectively. Hence, this equation is called the **slope-intercept form**.

The official name for this type of relationship is **linear function**. The equations that describe it are called **linear equations**.

5. In Questions 2–4 you investigated graphs of members of the $y = mx + b$ family. The control numbers m and b are called the slope and y-intercept, respectively. Do you think that slope and y-intercept are descriptive names for m and b ? Why or why not?

Use **Figure 3.8** to answer Questions 6–9.

6. Figure 3.8 shows graphs of four linear equations. The line corresponding to $y = \frac{1}{2}x + 1$ has already been labeled with its equation.
 - a) For this line, what is the value of y when $x = 0$? How can you read this information from the equation?

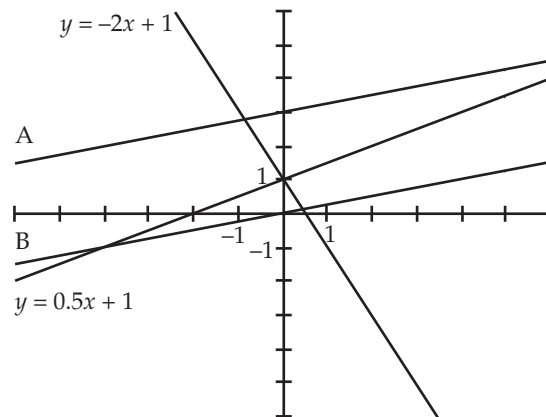


Figure 3.8.
Graphs of four lines.

- b) The point $(2, 3)$ lies on this line. (Look at the graph and check that $(2, 2)$ is on this line.) Suppose you change the value of x from 2 to 4. In this case $\Delta x = 4 - 2 = 2$. Find Δy , the corresponding change in y .
- c) Use your values for Δx and Δy from (b) to find the value of $\Delta y/\Delta x$. How is this ratio related to the equation of this line?
- d) The point $(4, 3)$ lies on this line because $\frac{1}{2}(4) + 1 = 3$. Find the coordinates of another point on the line.
- e) Find Δx and Δy for the two points in (d). Then calculate the value of $\Delta y/\Delta x$. Compare your answer to (c).

7. Next, look at the line corresponding to $y = -2x + 1$.

- a) Suppose you start with $x = 0$, and then change the value of x from 0 to 3. (So, $\Delta x = 3$.) What are the corresponding values for y ?
- b) Based on your y values from (a), what is Δy ?
- c) What is the value of $\Delta y/\Delta x$? How is this ratio related to the equation $y = -2x + 1$?
- d) Select two other values for x . Use the equation of this line to find the corresponding y values.
- e) Using your points from (d), what is the value of $\Delta y/\Delta x$? Compare your answer to (c).

In Questions 6 and 7, you should have noticed that the value of $\Delta y/\Delta x$ did not depend on which points you selected on the line. Also, its value agreed with the control number m , the slope of the line.

8. a) What is the slope of line B? How did you find it?
 b) What is the equation of line B? Explain how you know.
9. Find an equation describing line A. Explain how you got your answer.

In Questions 8 and 9, you found equations that described linear relationships between y and x . In each case, you could write your equation in slope-intercept form.

The statement “ y is a linear function of x ,” means that the relationship between x and y can be expressed as

$$y = mx + b,$$

where b is the value of y when $x = 0$ (the y -intercept) and

$$m = \frac{\Delta y}{\Delta x} \text{ is the slope.}$$

Graphs of linear functions are straight lines.

TAKE NOTE

When you compute the value of $\Delta y/\Delta x$, be careful to perform the subtraction in the correct order. If the coordinates of two points on the line are (x_1, y_1) and (x_2, y_2)

$$\text{then } \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

10. a) You can think of a linear function as a coding process. Each x -value that you put into this function, produces a single “coded x -value” for y . Draw an arrow diagram that represents the function $y = mx + b$.
- b) Can you think of any values for m or b that would make a really bad coding process? Explain.

Activity Summary

In this activity, you:

- ♦ discovered how the values of m and b affect the graphs of linear functions, $y = mx + b$.
- ♦ found equations of lines given their graphs.

DISCUSSION/REFLECTION

Suppose your model is $y = 3x + 5$. You gather more data. Then, you graph your model and plot the data. Figure 3.9 shows how things look.

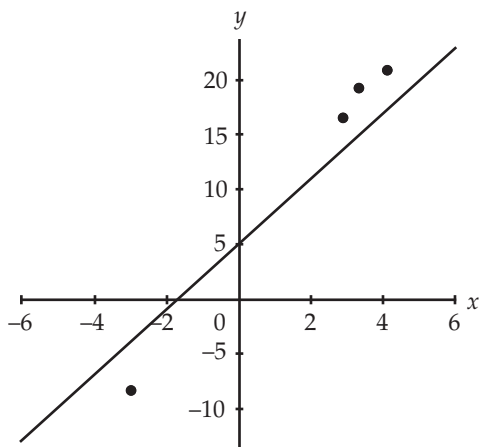


Figure 3.9.
A graph of a model and some new data.

1. Suppose you want to adjust your model so that its graph is closer to the new data points. You plan to change only one of the control numbers in your model. Would it be better to change m or b ? Why?
2. You’ve decided which control number you want to change. Should you make this number larger or smaller? Explain.

This activity gives you practice graphing a line from its equation and finding an equation from a graph. You will apply what you have learned about lines to a real-world situation. You will extend your knowledge of linear equations to include the $y = k + m(x - h)$ family.

1. Answer the following questions *without* graphing the equations.
 - a) Which graph is steeper, the graph of $y = 3.48x + 20$ or the graph of $y = 5.78x + 5$? How do you know?
 - b) Which graph crosses the y -axis at 30, the graph of $y = 30x + 15$ or $y = 15x + 30$? How do you know?
 - c) Which graph slants downward as the x -values increase, $y = \frac{1}{2}x + 15$ or $y = -2x + 5$? How do you know?

TAKE NOTE

When mathematicians say, "the graph is a line," they mean a *straight* line.

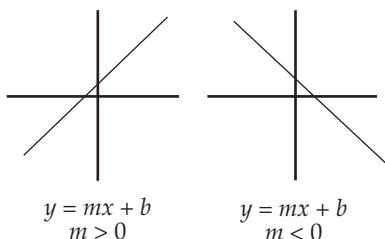


Figure 3.10. Generic graphs of members of the $y = mx + b$ family.

The equations in Question 1 are all linear equations written in slope-intercept form. Their graphs are lines that resemble one of the graphs in **Figure 3.10**. Recall that if you know two points on a line, you can calculate its slope using the ratio $m = \Delta y / \Delta x$.

2. Find the slope of each line described below. Show your calculations.
 - a) The line that passes through the points (1, 4) and (7, 6).
 - b) The line that passes through the points (2, 9) and (5, 4).
3. For each of the graphs in **Figures 3.11–3.14** write an equation in the form $y = mx + b$ that could represent the graph. Explain how you found your values for m and b . (Warning: Check the graphs to see if their slopes are positive or negative.)

TAKE NOTE

It doesn't matter whether you write $y = mx + b$ or $y = b + mx$. Either way, you will get the same graph. That's because these two equations are algebraically equivalent.

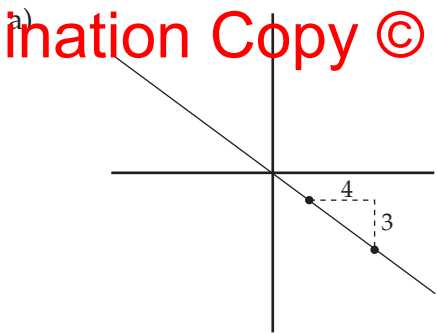


Figure 3.11.
Graph of a linear equation.

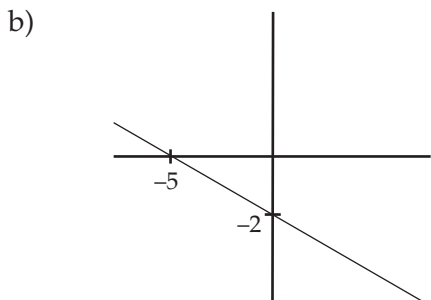


Figure 3.12.
Graph of a linear equation.

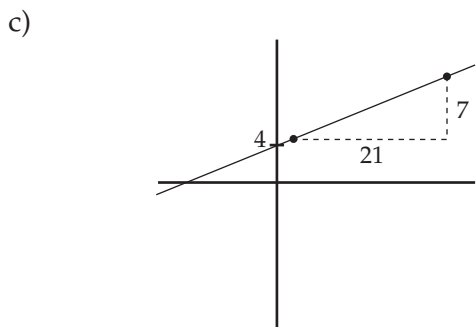


Figure 3.13.
Graph of a linear equation.

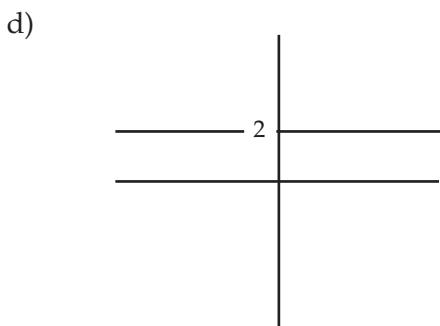


Figure 3.14.
Graph of a linear equation.

4. Draw the graphs of $y = 2 + 3x$ and $y = 3 - x$ using the following hints.
- **Hint for graphing $y = 2 + 3x$:** Since $b = 2$, mark a point on the y -axis at 2. The slope is $m = 3 = \frac{3}{1}$. Starting from the y -intercept, move up 3 units and across 1 unit. Then mark a second point. To mark more

points, move up 3 units and across 1 unit, up 3 and across 1, and so forth. Draw your line through the points you have marked (see Figure 3.15).

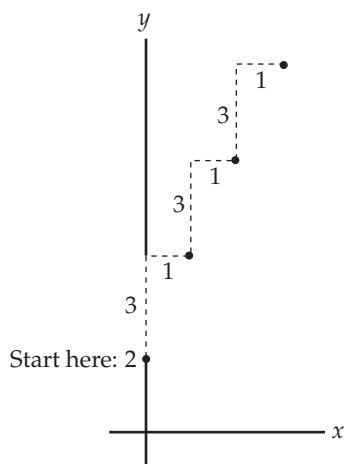
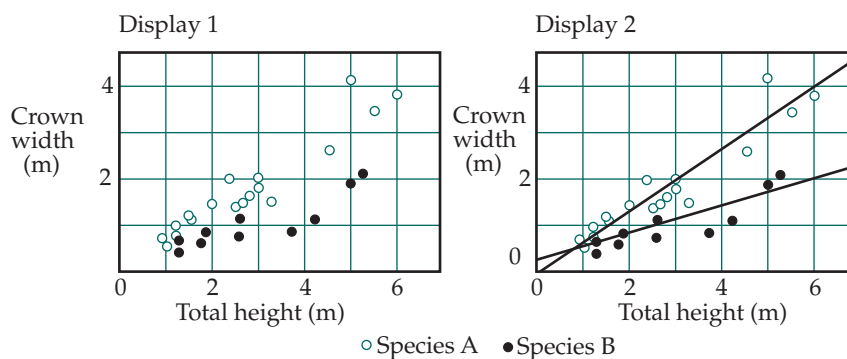


Figure 3.15.
Moving up 3
and across 1.

- **Hint for graphing $y = 3 - x$:** Since $b = 3$, mark $y = 3$ on the y -axis. The slope is $m = -1 = \frac{-1}{1}$. Mark additional points as follows: starting from the y -intercept, move down 1 unit and across 1 unit, then down 1 and across 1, and so on. Draw your line.
5. “Understory” trees are the short trees among much taller trees in a forest or jungle. Their growth is stunted because of the thick vegetation above them. Although understory trees are shorter than other trees, their crowns can be very wide.

Figure 3.16.
Understory trees
in a forest.



Biologists studied two species of understory trees. Display 1 in Figure 3.16 shows a plot of their data. To sharpen the relationship between height and width, they drew lines that they thought described the general pattern of the data for each species of tree (see Figure 3.16, Display 2.)

- a) For each species, predict the crown width when the tree height is 4 meters.

- b) For each species, predict the tree height when the crown width is 2 meters.
- c) Which of the two lines in Display 2 can be described by a linear equation from the $y = mx$ family? How can you tell? What is the value for m (approximately)? How did you determine m 's value?
- d) The other line can be described by a linear equation from the $y = mx + b$ family. Determine the equation for this line.
- e) In your equation for Species B, interpret the control numbers in the context this problem. In other words, what does m mean in this context? What, if anything, does b mean?

Up to this point the linear equations you have worked with have been written in slope-intercept form, $y = mx + b$. Suppose you choose $b = 3$. Then graphs of all members of the $y = mx + 3$ family pass through the point $(0, 3)$. (Why?) But suppose that you wanted your lines to pass through the point $(4, 3)$ instead. What equations would you use to describe these lines? The key to the answer is contained in Question 6.

6. Check that the following lines all pass through the point $(4, 3)$. In other words, show that you get $y = 3$ when you substitute 4 in for x .
- a) $y - 3 = 2(x - 4)$
- b) $y = 5(x - 4) + 3$
- c) $y - 3 = -2(x - 4)$
- d) $y = 2(x - 4) + 3$
- e) $y = m(x - 4) + 3$. (Even though you don't know the value of m , you can still check that $y = 3$ when $x = 4$.)
7. Two of the equations in Question 6 are equivalent. Which two? How did you decide?

The equations in Questions 6 and 7 all belong to (or are closely related to) the $y = k + m(x - h)$ family. You'll learn more about this family in Questions 8–11.

8. **Figure 3.17** shows a plot of some data. Its center is marked with an "X" at $(10, 20)$. A line is drawn through the X in the general direction of the pattern of dots.

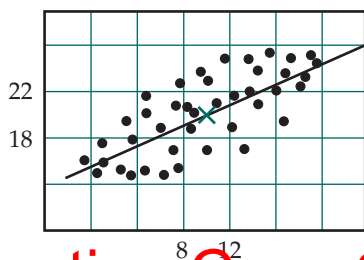


Figure 3.17.
Data with X at center.

TAKE NOTE

If you know the slope of a line, m , and its y-intercept, b , you can use the slope-intercept form, $y = mx + b$, to find the line's equation.

TAKE NOTE

Coordinates of points that you find from a graph are only approximate. Hence any equation you create based on these points will also be approximate. If you compare your equation from 8(c) to the equations of other students they may not be algebraically equivalent.

- a) Explain why the following statement is true: The graph of any linear equation from the $y = m(x - 10) + 20$ family will pass through the X in this plot.
- b) Approximately what is the slope of the line? Explain how you arrived at your answer.
- c) Determine an equation for the line.
- d) Find an equation from the $y = mx + b$ family that produces the same graph as your equation in (c). (Hint: Use the distributive law.)
9. The average girl in the United States is 102 cm tall at age 4. From then until age 13 she grows, on average, 6 cm every year.
- a) Draw a graph of this model. Start by plotting the point (4, 102). Then use the fact that she grows 6 cm every year to locate other points on the graph. Then, draw your graph.
- b) Write an equation that describes your graph.

In Questions 8 and 9, you had to find an equation for a line given a point on the line and the line's slope. Linear equations written in **point-slope form** are just what you need in situations like these.

Suppose that the point (h, k) lies on a line that has slope m . Then, you can use the following equation to describe this line:

$$y = m(x - h) + k$$

This form of linear equation is called the **point-slope form**.

10. a) Draw the graph of a line that has slope $\frac{1}{2}$ and passes through the point (1, 4). Explain the process you used to draw your line.
- b) Write an equation in point-slope form that describes this line.
11. Find equations in point-slope form that could describe each of the lines in **Figures 3.18 and 3.19**. Explain how you found the equation.

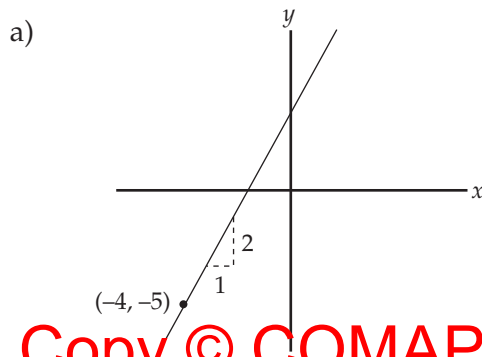


Figure 3.18.
Graph of a linear equation.

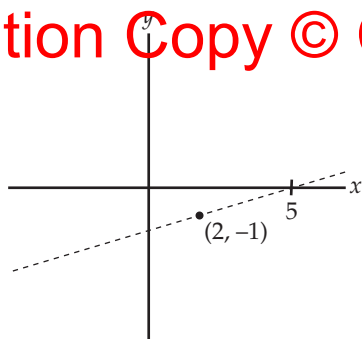


Figure 3.19.
Graph of a linear equation.

12. A line passes through the points (2, 8) and (5, 4). Juan said that the equation of this line was $y = 8 + \frac{-4}{3}(x - 2)$. Katie said it was $y = 4 + \frac{-4}{3}(x - 5)$.
- Graph both equations. What is true about the two graphs?
 - Use the distributive law to re-express Juan's equation in slope-intercept form.
 - Use the distributive law to re-express Katie's equation in slope-intercept form.
 - Whose answer is correct, Juan's or Katie's? Explain.

TAKE NOTE

Recall that two equations are equivalent if they have the same solution. In the case of Question 12 the two equations are equivalent if they produce the set of points; in other words, if their graphs are the same. If the graphs don't match, then the equations are not equivalent.

Activity 3.3: From Head to Toes

In this activity you apply what you have learned about linear equations to develop models based on real data. You look at the residual errors associated with your models. Based on the residual errors, you may decide to adjust your models.

FYI

Shortly after birth, the length of a baby's body and the circumference of its head are measured and recorded. These measurements can provide doctors with clues to future health problems. For example, a large head compared to the body might indicate fluid in the brain. A small head compared to the body might indicate an under-developed brain.

Doctors look at many factors when assessing the health of a newborn. One such factor is the size of the baby's head compared to the length of its body. However, body-length and head-circumference measurements vary from baby to baby. So, how do doctors determine what is normal for healthy babies?

Length (in.)	Head circumference (in.)
$18\frac{1}{4}$	$12\frac{3}{4}$
$18\frac{3}{4}$	13
$19\frac{1}{4}$	$13\frac{1}{4}$
20	$13\frac{3}{4}$
$20\frac{1}{2}$	14
21	$14\frac{1}{2}$
$21\frac{1}{2}$	$14\frac{3}{4}$

Figure 3.20.
Boys at birth.

Length (in.)	Head circumference (in.)
$17\frac{3}{4}$	$12\frac{3}{4}$
$18\frac{1}{4}$	13
19	$13\frac{1}{4}$
$19\frac{3}{4}$	$13\frac{1}{2}$
20	$13\frac{3}{4}$
$20\frac{1}{2}$	14
$20\frac{3}{4}$	$14\frac{1}{4}$

Figure 3.21.
Girls at birth.

The data in Figures 3.20 and 3.21 give you an idea of what are typical measurements for small to large babies. (These data come from a reference book for pediatricians.) Your task will be to use these data to develop models (equations) for predicting a baby's head circumference given its body length.

1. Why do you think the data are separated by gender?
2. Suppose you have developed a model that predicts a baby's head circumference from its body length. Which is the explanatory variable? Which is the response variable? Justify your answer.

Your task in Questions 3–5 will be to find two linear equations, one for each gender, that describe the relationship between body length and head circumference. Later you can look for patterns in residual errors and decide if you need to adjust your equations.

Divide the work among the members of your group: half should work on the model for boys and the other half on the model for girls.

3. a) On graph paper, draw a set of axes. Label each axis with the appropriate variable and units of measurement. Mark each axis with a reasonable scale. Remember to insert a zigzag if you break the scale near zero.



Remember the explanatory variable goes on the horizontal axis. The response variable goes on the vertical axis.

A graph of ordered pairs of data is called a scatter plot.

- b) On the axes you drew for (a), make scatter plots of the data from Figures 3.20 and 3.21. (In other words, plot a point for each ordered pair, (body length, head circumference).)
4. a) Use a ruler to draw a straight line that closely resembles the pattern of the dots in the scatter plot. This is called fitting a line to the data.
 - b) What is the approximate slope of your line? How did you determine the slope?
 - c) What does the slope tell you about babies?
 - d) On your line, determine the coordinates of one point that lies somewhere near the center of your scatter plot. (Your point does not need to be one of the data points.) What is your point?



The dots in a scatter plot rarely fall exactly on a line. So, don't try to draw a line through all the points. Instead, find a line that slants in about the same direction as the pattern of dots. Also, find a line that is close to the dots. Below are two lines drawn on the same scatter plot. The line in **Figure 3.22** fits the data fairly well. The line in **Figure 3.23** does not.

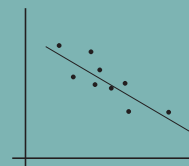


Figure 3.22. The line follows the pattern of the data. The dots are close to the line.

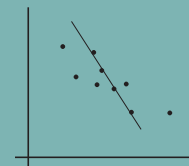


Figure 3.23. The line does not follow the pattern of the data. The line goes through 4 dots. But some dots are not close to the line.

e) Now use your answers to (b) and (d) to find an equation in point-slope form that describes your line.

5. a) What member of the $y = mx + b$ family is equivalent to your equation from 4(e)? Explain how you got your answer.
 b) Does the value of b make sense in this context? Explain.

TAKE NOTE

You may have chosen different letters than x and y to represent body length and head circumference. The form $y = mx + b$ does not mean you have to use the letters x and y for your variables. For example, if you used c for head circumference and l for body length, write your model in the form $c = ml + b$.

6. a) Enter the data (either for boys or for girls) into your calculator.
 b) Set your window to match the scaling on the graph that you drew by hand. Make a graph of the line and a scatter plot of the data. Compare the display on your screen with the one that you drew by hand. (If the line is not where you intended, check to see that you have determined the equation of your hand-drawn line correctly.)

Length (in.)		Predicted head circumference (in.)	Actual head circumference (in.)		Residual errors (in.)
Boys	Girls		Boys	Girls	
$18\frac{1}{4}$	$17\frac{3}{4}$		$12\frac{3}{4}$	$12\frac{3}{4}$	
$18\frac{3}{4}$	$18\frac{1}{4}$		13	13	
$19\frac{1}{4}$	19		$13\frac{1}{4}$	$13\frac{1}{4}$	
20	$19\frac{3}{4}$		$13\frac{3}{4}$	$13\frac{1}{2}$	
$20\frac{1}{2}$	20		14	$13\frac{3}{4}$	
21	$20\frac{1}{2}$		$14\frac{1}{2}$	14	
$21\frac{1}{2}$	$20\frac{3}{4}$		$14\frac{3}{4}$	$14\frac{1}{4}$	

Figure 3.22. Predicted head circumference and residual errors from $y = mx + b$ model.

7. Copy the table from **Figure 3.22** (for either the boys' or girls' data, not both).
 a) Use your equation from Question 5 to complete the entries for the Predicted-head circumference column. (Hint: To accomplish this task quickly, apply your equation to the length-data list in your calculator. See Handout H3.4 for instructions.)
 b) Compute the residual errors. (In other words, calculate actual head circumference minus predicted head circumference.)



You want a model where the residual errors are *nearly* equally divided between positive and negative values. It is not necessary that the number of positive and negative values is exactly equal.

8. a) If a residual error is positive, what does that tell you about your prediction?
 - b) What if an error is negative?
 - c) What if an error is 0?
9. a) Return to your residual errors from 6(b). Are there about the same number of positive and negative residual errors?
 - b) If your answer to (a) is yes, skip to (c). If your answer to (a) is no, try to improve your model by adjusting the values for m and b . Then, calculate the residual errors for your new model. Check to see if there are about the same number of positive and negative residuals.
 - c) Are the errors small in comparison to the head circumferences you are trying to predict? Explain.
10. a) What is the absolute value of the worst residual error?
 - b) Suppose the absolute value of the worst error was small relative to the values you were predicting. What would that tell you about your model?
11. a) What is the average of the residual errors? (To find the average, sum the residuals and divide by 7.)
 - b) What value for the average residual error would be best? Explain.
12. Bring the two halves of your group together. Graph the two relationships, the one for the boys and the one for the girls, on the same set of axes. Do the relationships between head circumference and body length appear to be different for boy babies and girl babies? Explain.

In this activity, you graphed lines that followed the pattern of the data in your scatter plots. Your models, the equations of these lines, are called **linear models**. What makes fitting a linear model difficult is that real data rarely fall exactly on a line. In later lessons you will learn one method that statisticians use to pick the “best-fitting line.” This method will be very useful when your data do not fall as close to a line as the data in this activity.

Activity Summary

During this activity, you:

- ♦ made scatter plots of the data.
- ♦ found linear models that described patterns in your scatter plots.
- ♦ looked at the residual errors associated with your models. You may have adjusted your model after looking at the residual errors.

DISCUSSION/REFLECTION

The models you calculated for the relationship between head circumference and body length were members of the $y = mx + b$ family with $b \neq 0$. Sometimes doctors prefer simpler models, ones from the $y = mx$ family.

1. Why might a model of the form $y = mx$ be more useful to doctors than a model of the form $y = mx + b$?
2. Below are two more models for predicting head circumference from height:

Boys' model: $y = 0.69x$

Girls' model: $y = 0.70x$

How would you use residual errors to decide which models are better, these models from the $y = mx$ family or the ones that you found in this activity?

This assignment provides practice with finding and graphing linear equations. In addition, you solve problems by manipulating linear equations symbolically.

1. a) Locate two points on each of the lines in **Figure 3.23**. What are the coordinates of your points?
- b) Using your points from (a), write two possible equations in point-slope form for each of the lines. (You may want to use your calculator to check that your equations produce the same graphs as Figure 3.23.)
- c) Find equations from the $y = mx + b$ family that are equivalent to your equations in (b).

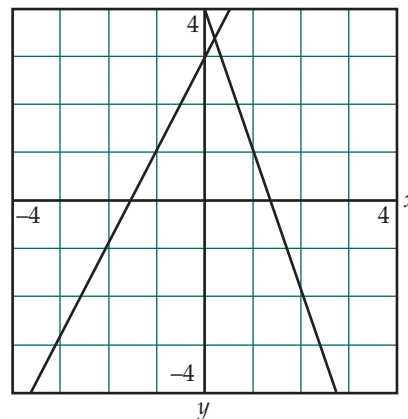


Figure 3.23. Two lines.

A set of axes uses **square scaling** if equal distances on the axes represent equal intervals. For example, in **Figure 3.24**, the distance between 0 and 1 on the horizontal axis is the same as the distance between 0 and 1 on the vertical axis. Use zoom square on your calculator to create axes with square scaling.



Figure 3.24. Axes with square scaling.

2. On a single set of axes, draw graphs of equations (a)–(d). Use square scaling on your axes. What shape do the lines appear to enclose?
 - a) $y - 4 = -2(x - 4)$
 - b) $y - 4 = \frac{1}{2}(x - 4)$
 - c) $y = -2(x - 3) + 1$
 - d) $y = \frac{1}{2}(x - 3) + 1$
3. Make up and solve a problem similar to the one you just answered in Question 2. (Make sure the equations that you specify enclose some shape.)

4. On December 18, 1994, three amateur spelunkers (cave explorers) stumbled across a cave in France, now known as Chauvet Cave. Inside the cave they found ancient cave paintings. In later explorations human footprints were found. According to prehistorian Michel-Alain Garcia the footprints belonged to a boy who was about four-and-a-half-feet tall and lived between 20,000 and 30,000 years ago.

TAKE NOTE

If you measure your feet, most likely one foot will be longer than the other. The model given in Question 4(a) tells you to use the measurement from the larger foot.

- a) Scientists have used simple models to predict height from footprints since the mid-1800s. One model, still in use today, predicts height by dividing the maximum foot length by 0.15. Write a formula that describes this model.
- b) The model in (a) can be expressed as a member of the $y = mx$ family. What is the approximate value of m ?
- c) Using this model, predict the length of the larger footprints in Chauvet Cave. Give your answer in centimeters. (Remember 2.54 cm = 1 in.)
- d) Suppose that another footprint was found and it measured 1 cm more than the one discussed by Garcia. By how much would you increase your estimate for height? Explain how you got your answer.

5. Anthropologists have refined early models for estimating a person's height from the length of a footprint. One revision suggests using different models depending on whether the footprint was from the right foot or from the left.

right foot: $H = 3.641L + 72.92$

left foot: $H = 4.229L + 56.49$,

where all measurements are in cm.

- a) Suppose a footprint from a right foot measured 22 cm. Predict the person's height.
 - b) Suppose a 22-cm footprint was from a left foot. Predict the person's height.
 - c) Both the right-foot and left-foot models are from the $y = mx + b$ family. For each of the models interpret the meaning of slope, m , in the height-footprint context.
 - d) What does b mean for each of these models in the height-footprint context?
6. a) Solve the right-foot equation in Question 5 for L . (Think of the right-foot equation as a coding process that codes foot-length values into height values. Solve for L by decoding.)
- b) Now solve the left-foot equation in Question 5 for L .

- c) Your answers to parts (a) and (b) are linear equations in L and H . Find equivalent equations that have the form $L = mH + b$ (slope intercept form). Explain how you got your answers.
- d) Interpret the meaning of the slopes of your equations in (c).
- e) Suppose a person was 153 cm tall. Use your equations from (a) and (b) or your equations from (c) to predict the lengths of the person's right and left footprints. How much longer is the larger foot than the shorter foot?

Questions 7–10 refer to the IQ data in **Figure 3.25**.

7. According to scores on intelligence tests given since the 1930s, Americans are getting smarter. Figure 3.25 provides data on the average IQ scores of Americans.
- a) Calculations for a model will be easier if you let x represent the years since 1932. Hence, $x = 0$ represents 1932. What x -value represents 1947? What x -value represents 1952? (Notice that the years are not equally spaced.)
- b) Make a scatter plot of average IQ versus x . That means use the vertical axis for average IQ and the horizontal axis for x . (Be sure to draw a zigzag if you break the scale between 0 and another number.)
- c) Draw a line on your scatter plot that you think best fits the pattern of the data. What criterion did you use in selecting your line?
- d) Write an equation for the line that you drew in part (c). How did you determine your equation? If your equation is not in $y = mx + b$ form, find an equivalent equation that is in this form.
8. a) Calculate the residuals for the model you found in 7(d).
 b) Based on your answer to part (a), do you think your model does a good job describing these data? Explain why or why not.
 c) If you are unhappy with your model, find a better one.
9. You are considered to be a genius if you have an IQ over 140.
 a) Use your model from 7(d) or 8(d) to predict the year in which the average IQ will reach genius-level.
 b) Do you believe this? Explain why or why not.
10. a) Suppose you decided to use the years as the explanatory variable in your IQ-score model. Determine an equation for a line that passes through the first and last data points: (1932, 100) and (1997, 118).

Year	Average IQ
1932	100
1947	106
1952	108
1972	112
1978	113
1997	118

Figure 3.25.
Average IQ scores from 1932 to 1997.



When you use the term **versus**, the response variable, y , comes first followed by the explanatory variable, x . Hence, you should write y versus x when you are referring to the relationship between x and y .

- b) Find an equation in slope-intercept form that is equivalent to your equation from part (a).
- c) Compare this model to the one you found for 7(d). Which model would you prefer to use? Why?