Math 270 Basic Discrete Math
Practice Test 3
Sections 5.1, 5.2, 5.3, 5.4, 5.6, 5.7

Name: (Please Print) Solutions

Directions: Answer the problems below. You may use scientific (non-graphing) calculators, but no other electronic devices. Show all work.

1. Prove, using mathematical induction, that for all integers $n \geq 1$,

$$3 + 7 + 11 + \cdots + (4n - 1) = 2n(n + 1) - n.$$

Proof: Let $P(n) =$ "$3+7+11+\cdots+(4n-1) = 2n(n+1)-n$."

We prove $P(n)$ holds for all $n \geq 1$ by induction.

Base case ($n=1$): We show $P(1)$ holds: the LHS of $P(1)$ is 3, whereas the RHS of $P(1)$ is $2\cdot 1(1+1)-1 = 4-1=3$, so LHS = RHS and $P(1)$ holds.

Inductive step: Let $k \geq 1$ be arbitrary and suppose $P(k)$ holds, so

$$3+7+11+\cdots+(4k-1) = 2k(k+1)-k.$$

We must show $P(k+1)$ holds, i.e. that

$$3+7+11+\cdots+(4k-1)+(4(k+1)-1) = 2(k+1)(k+2)-(k+1).$$

Starting with the LHS,

$$\begin{aligned}
3+7+11+\cdots+(4k-1)+(4(k+1)-1) &= \left(3+7+\cdots+(4k-1)\right)+(4(k+1)-1) \\
&= \left(2k(k+1)-k\right)+(4(k+1)-1) \qquad \text{(By I.H.)} \\
&= 2k(k+1)-k+2\cdot 2(k+1)-1 \\
&= 2(k+2)(k+1)-(k+1) \\
&= 2(k+1)(k+2)-(k+1).
\end{aligned}$$

So $P(k+1)$ holds, proving the inductive step.

Since the inductive step and base case hold, $P(n)$ is true for all integers $n \geq 1$. ∎

2. Let $a_1, a_2, a_3, \ldots$ be the sequence defined recursively as follows:

$$a_1 = 1,\ a_2 = 20, \text{ and for all } k \geq 3,\ a_k = 5a_{k-1} + 6a_{k-2}.$$

Use strong induction to prove that for all integers $n \geq 1$, $a_n \leq 6^n$.

<u>Proof</u>: Let $P(n)$ = "$a_n \leq 6^n$". We show $P(n)$ holds for all $n \geq 1$ by strong induction.

<u>Base Cases ($n=1,2$)</u>: Since $a_1 = 1 \leq 6 = 6^1$ and $a_2 = 20 \leq 36 = 6^2$, both $P(1)$ and $P(2)$ hold.

<u>Inductive Step</u>: Let $k \geq 2$ be arbitrary and suppose $P(1), P(2), \ldots, P(k)$ all hold. We want to show that $P(k+1)$ holds, i.e., that $a_{k+1} \leq 6^{k+1}$.

Since $k \geq 2$, $k+1 \geq 3$, so

$$\begin{aligned}
a_{k+1} &= 5a_k + 6a_{k-1} && \text{(By definition of } a_{k+1} \text{ as } k+1 \geq 3\text{)} \\
&\leq 5 \cdot 6^k + 6 \cdot 6^{k-1} && \text{(I.H.: } P(k-1) \text{ AND } P(k-2) \text{ hold)} \\
&= 5 \cdot 6^k + 6^k \\
&= 6 \cdot 6^k \\
&= 6^{k+1},
\end{aligned}$$

so $P(k+1)$ holds. Therefore the inductive step holds, and since the base cases hold, $P(n)$ is true for all $n \geq 1$ by strong induction. ∎

3. Provide short responses for parts a.-d. below.

a. Calculate each of the following:

i. $\prod_{i=1}^{4}(2i) = 2\cdot 4\cdot 6\cdot 8 = 384$

ii. $\sum_{i=1}^{4}(2i-1) = 1+3+5+7 = 16$

iii. $\frac{4!}{2!} = \frac{4\cdot 3\cdot 2\cdot 1}{2\cdot 1} = 4\cdot 3 = 12$

iv. $\binom{6}{2} = \frac{6!}{2!\,4!} = \frac{6\cdot 5}{2\cdot 1} = 15$

v. $\binom{6}{4} = \frac{6!}{4!\,2!} = 15$

vi. $\binom{6}{0} = \frac{\cancel{6!}}{0!\,\cancel{6!}} = \frac{1}{0!} = \boxed{1.}$

$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5$

b. Suppose the sequence $a_1, a_2, a_3, \ldots$ begins with the terms $8, -27, 64, -125, 216, \ldots$. Find an explicit formula for a_n.

$2^3 \quad -3^3 \quad 4^3 \quad -5^3 \quad 6^3$

$$a_n = (-1)^{n+1}(n+1)^3$$

c. Write the product $(1-t)(1-2t^2)(1-3t^3)(1-4t^4)$ using product notation.

$i=1 \quad i=2 \quad i=3 \quad i=4$

$$= \prod_{i=1}^{4}\left(1 - i\cdot t^i\right).$$

d. Transform the sum $\sum_{j=3}^{n+1} \frac{j^2-1}{n-j+2}$ by making the change of variable $i = j-2$.

$i = j-2$: $j=3 \Rightarrow i=1$

$j=n+1 \Rightarrow i=n-1$

$j = i+2$

$$\sum_{i=1}^{n-1} \frac{(i+2)^2-1}{n-(i+2)+2} = \boxed{\sum_{i=1}^{n-1} \frac{i^2+4i+3}{n-i}}$$

4. Find explicit formulas for the following recurrence relations. (You do *not* need to prove your answers are correct.) Simplify your answers as much as possible: for full credit your answers should include no summation or product notation.

a. $a_1 = 1$, $a_k = a_{k-1} + 2$ for all $k \geq 2$.

$$a_2 = a_1 + 2 = 1 + 2$$
$$a_3 = a_2 + 2 = (1+2)+2$$
$$a_4 = a_3 + 2 = ((1+2)+2)+2 = 1 + 3\cdot 2 = 4\cdot 2 - 1$$

$$\boxed{a_n = 2n - 1}$$

b. $b_1 = 2$, $b_k = k \cdot b_{k-1}$ for all $k \geq 2$.

$$b_2 = 2\cdot b_1 = 2\cdot 2$$
$$b_3 = 3\cdot b_2 = 3\cdot 2\cdot 2$$
$$b_4 = 4\cdot b_3 = 4\cdot 3\cdot 2\cdot 2 = \underbrace{4\cdot 3\cdot 2\cdot 1}\cdot 2 = 2\cdot 4!$$

$$\boxed{b_n = 2\cdot n!}$$

c. $c_1 = 0$, $c_k = c_{k-1} + 2k$ for all $k \geq 2$.

$$c_2 = c_1 + 2(2) = 2(2)$$
$$c_3 = c_2 + 2(3) = 2(2) + 2(3)$$
$$c_4 = c_3 + 2(4) = 2(2) + 2(3) + 2(4) = 2(2+3+4)$$

$$c_n = 2(2+3+\cdots+n) = 2(1+2+\cdots+n - 1) = 2\left(\frac{n(n+1)}{2} - 1\right) = n(n+1) - 2,$$

so $\boxed{c_n = n^2 + n - 2}$

d. $d_1 = 1$, $d_k = 2d_{k-1} + 1$ for all $k \geq 2$.

$$d_2 = 2d_1 + 1 = 2(1) + 1 = 2 + 1$$
$$d_3 = 2d_2 + 1 = 2(2(1)+1) + 1 = 2^2 + 2 + 1$$
$$d_4 = 2d_3 + 1 = 2(2(2(1)+1)+1) + 1 = 2^3 + 2^2 + 2 + 1$$

So $d_n = \underbrace{1 + 2 + 2^2 + \cdots + 2^{n-1}}_{\text{geometric sum}} = \dfrac{2^n - 1}{2 - 1}$ $\therefore$ $\boxed{d_n = 2^n - 1.}$