Math 270 Basic Discrete Math
Practice Test 4
Sections 5.8, 5.9, 6.1, 8.1, 8.2, 8.3

Name: (Please Print) Solutions

Directions: Answer the problems below. You may use scientific (non-graphing) calculators, but no other electronic devices. Show all work.

1. Let $a_1, a_2, a_3, \ldots$ be the sequence defined recursively as follows:

$$a_1 = 9,\ a_2 = 21, \text{ and for all } k \geq 3,\ a_k = 5a_{k-1} - 6a_{k-2}.$$

Find an explicit formula for a_n.

The characteristic equation is $t^2 - 5t + 6 = 0$

$(t-2)(t-3) = 0$,

so roots are $t = 2, 3$. hence

$a_n = C \cdot 2^n + D \cdot 3^n$ for some C, D.

$a_1 = 9 = C \cdot 2 + D \cdot 3 \Rightarrow \underbrace{2C + 3D = 9}_{EQ_1}$

$a_2 = 21 = C \cdot 2^2 + D \cdot 3^2 \Rightarrow \underbrace{4C + 9D = 21}_{EQ_2}$

By subtracting $EQ_2 - 2(EQ_1)$ we get $3D = 21 - 18 = 3$,

so $D = 1$, and hence $C = \frac{1}{2}(9-3) = 3$.

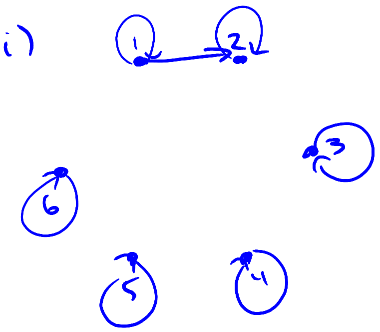
Therefore,

$$\boxed{a_n = 3 \cdot 2^n + 3^n}$$

2. In this problem, let $A = \{1, 2, 3, 4, 5, 6\}$. Answer a.-c. below.

a. Describe a relation R on A which is reflexive but is not symmetric by (i) drawing the digraph for R and (ii) listing the elements of R in set-roster notation.

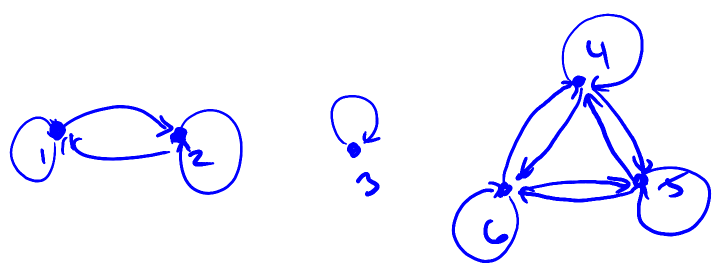
(i)



(ii)

$R = \{(1,1), (1,2), (2,2), (3,3), (4,4), (5,5), (6,6)\}$.

b. Draw the digraph for an equivalence relation S on A which has three distinct equivalence classes: $\{1, 2\}$, $\{3\}$, and $\{4, 5, 6\}$. (You only need to draw the digraph.)



c. Let T be the equivalence relation on A given by

$$x\,T\,y \quad \Leftrightarrow \quad 3|(x^2 - y^2).$$

What are T's distinct equivalence classes?

For [1]: $1T1$, $1T2$, $1T4$, $1T5$, $1 \not{T} 3$, $1 \not{T} 6$;
so $[1] = \{1, 2, 4, 5\}$

Next [3]: $3T3$, $3T6$, so $[3] = \{3, 6\}$; no elements of A left.

The distinct equivalence classes are $\boxed{\{1, 2, 4, 5\}, \{3, 6\}}$.

3. Answer parts a. and b. below.

a. Let the universe $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and let

$$A = \{1, 2, 3, 4\}, \quad B = \{2, 4, 6, 8, 10\}, \quad \text{and} \quad C = \{3, 6, 9\}.$$

Calculate each of the following:

i. $A \cup B =$ $\{1,2,3,4,6,8,10\}$

ii. $B \cap C =$ $\{6\}$

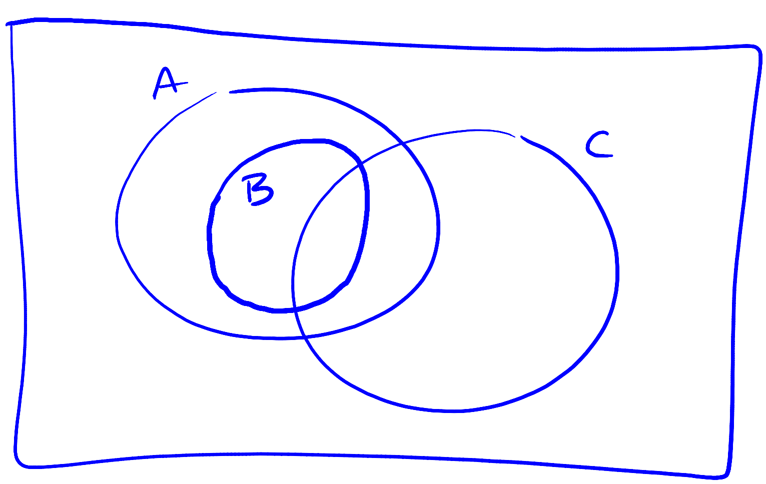
iii. $(A \cup B) - C =$ $\{1,2,4,8,10\}$

iv. $A \cap B \cap C =$ $\emptyset$

v. $(A \cup B)^c =$ $\{5,7,9\}$

b. Draw a Venn diagram for three sets A, B, C which satisfy the following conditions:

$$A \subseteq B, \; A \cap C \neq \varnothing, \; B \cap C \neq \varnothing.$$



4. Define a set S of integers recursively as follows:

I. Base: $3 \in S$.

II. Recursion: if $k \in S$, then

II(a) $k + 6 \in S$

III. Restriction: Nothing is in S other than objects defined in I, II above.

Use structural induction to prove that every integer $n \in S$ is divisible by [redacted]3.

Proof. Let $P(n) =$ "$3 \mid n$" : we show $P(n)$ holds for all $n \in S$ by structural induction.

Base: The only element of the base is $n = 3$, and since $3 \mid 3$, $P(3)$ holds.

Inductive Step: Let $k \in S$ and suppose $P(k)$ holds, so $3 \mid k$. The recursion for S has only one rule, which applied to k gives us $k+6$.

Since $3 \mid k$, $k = 3q$ for some $q \in \mathbb{Z}$, so $k + 6 = 3q + 6 = 3(q+2)$.

As $q + 2 \in \mathbb{Z}$ by closure, $3 \mid (k+6)$ by definition, so $P(k+6)$ holds, completing the inductive step.

Since no other integers lie in S other than those obtained from the base and the recursion, $P(n)$ holds for all $n \in S$. ∎

5. Answer parts a.-c. below.

a. Define the sets

$$A = \{n \in \mathbb{Z} \mid n = 4a + 1 \text{ for some } a \in \mathbb{Z}\}, \text{ and}$$
$$B = \{m \in \mathbb{Z} \mid m = 4b + 3 \text{ for some } b \in \mathbb{Z}\}.$$

Are A and B disjoint? Why or why not?

Yes, they are disjoint. Suppose otherwise, that some n exists in $A \cap B$. Then since $n \in A$, $n \bmod 4 = 1$, and since $n \in B$, $n \bmod 4 = 3$. But this contradicts the uniqueness of the Quotient-Remainder Theorem, so no such n exists.

b. Express the power set $\mathscr{P}(\{1, 2, 3\})$ in set-roster notation.

$$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$