Math 270 Basic Discrete Math
Practice Test 5
Sections 9.2, 9.1, 9.3, 9.5, 9.6, 10.4

Name: (Please Print) Solutions

Directions: Answer the problems below. You may use scientific (non-graphing) calculators, but no other electronic devices. Show all work.

1. The code to a particular combination lock consists of an ordered selection of five numbers, each from 1 through 100.

a. How many different codes are possible?

Pick from first to last: $100^5 = 10{,}000{,}000{,}000$

b. How many codes have no repeated number?

Pick from first to last: $100 \cdot 99 \cdot 98 \cdot 97 \cdot 96 = 9{,}034{,}502{,}400$

c. How many codes have the same first and last number?

Pick from first to last: $100 \cdot 100 \cdot 100 \cdot 100 \cdot 1 = 100{,}000{,}000$

2. In the Commonwealth of Pennsylvania, car license plates consist of three upper-case letters (from A-Z) followed by four numerals (0-9), in the form *LLL-NNNN*. Order matters: *ABC-1234* is not the same as *BAC-4321*.

Answer parts a.-d. below. Your answer may include products, division, exponents, and factorials (such as, say, $2^2 \cdot 3 \cdot 4!$) if needed; you do not need to simplify.

a. How many different license plates does Pennsylvania have available?

Pick from first to last

$$26^3 \cdot 10^4$$

b. How many license plates have no repeated numerals? (Repeated letters are allowed.)

Pick from first to last

$$26^3 \cdot 10 \cdot 9 \cdot 8 \cdot 7$$

c. How many license plates have no repeated letters? (Repeated numerals are allowed.)

Pick om first to last

$$26 \cdot 25 \cdot 24 \cdot 10^4$$

d. What is the probability that a randomly chosen license plate has neither a repeated letter nor numeral?

$$\frac{26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{26^3 \cdot 10^4}.$$

3. Provide short answers for parts a.-d. below.

a. How many different 4-permutations are there of the elements of the set $\{a, b, c, d, e, f, g\}$?

$$7 \cdot 6 \cdot 5 \cdot 4 = 840$$

b. How many solutions are there to the equation $x+y+z = 10$ in nonnegative integers x, y, z?

= # 10-combinations of a 3-element set

$$= \binom{10+3-1}{10} = \boxed{\binom{12}{10}} \left(= 66\right).$$

c. Suppose a graph G has six vertices with degrees $1, 1, 3, 4, 4, 5$. Can G be a tree?

No: total degree is 18, so G has 9 edges,

and a tree on 6 vertices has only 5 edges.

d. In how many distinguishable ways can the letters of the word $\cancel{C}\cancel{O}MB\cancel{I}NAT\cancel{O}R\cancel{I}\cancel{C}S$ be arranged in order?

13 letters: 2 Cs, 2 Os, 2 Is, 1 each of M, B, N, A, T, R, S

$$\underset{C}{\binom{13}{2}}\underset{O}{\binom{11}{2}}\underset{I}{\binom{9}{2}}\underset{M}{\binom{7}{1}}\underset{B}{\binom{6}{1}}\underset{N}{\binom{5}{1}}\underset{A}{\binom{4}{1}}\underset{T}{\binom{3}{1}}\underset{R}{\binom{2}{1}}\underset{S}{\binom{1}{1}} = \frac{13!}{2!\cdot 2!\cdot 2!}.$$

4. Your instructor is designing an exam for Math 270. The exam will consist of five problems chosen from a collection of 15: five of these problems are 'easy', six problems are 'average', and four problems are 'hard'. Answer the problems below but you do not need to simplify: your answers may include products, division, sums, differences, exponents, factorials, and binomial coefficients (i.e. $\binom{n}{k}$) if appropriate.

a. How many possible exams can be constructed this way?

$$\binom{15}{5}$$

b. How many of these exams have exactly one hard problem?

$$\binom{4}{1} \cdot \binom{11}{4}$$

Pick hard problem

Pick other 4 problems from easy or average

c. How many of these exams consist solely of average problems?

$$\binom{6}{5}$$

d. Suppose your instructor has written two problems that are very, very similar: how many exams do not contain both of these problems (but may contain one)?

$$2\binom{15-2}{4} + \binom{15-2}{5}$$

Pick exactly one of these two

Now pick the other 4 problems from the rest

Don't use either problem

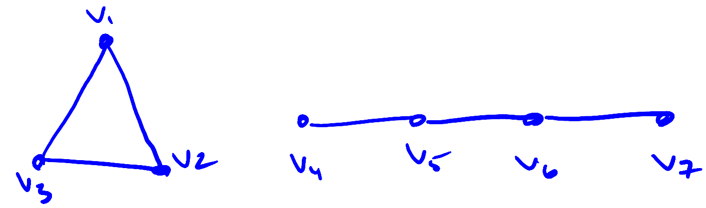
Also

$$\binom{15}{5} - \binom{15-2}{3}$$

Subtract the tests which have both from the total!

5. Answer parts a. and b. below.

a. Draw a graph G on 7 vertices with exactly 6 edges that is not a tree, and clearly state why it is not a tree.



G is not a tree because G is not connected.

b. Suppose T is a tree with exactly 5 internal vertices, of degrees $2, 3, 3, 4, 5$; all other vertices of T are leaves. How many leaves does T have?

Let $n = \#$ vertices of T and let $\ell = \#$ leaves of T,

so we know that $n = \ell + 5$.

The total degree of T must be $2(n-1) = 2\ell + 8$.

But each leaf contributes 1 to the total degree, so

it is also $2+3+3+4+5+\ell = 17+\ell$.

So

$$17 + \ell = 2\ell + 8,$$

so T has exactly $\ell = \boxed{9 \text{ leaves.}}$